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The Financial Immunization by a Sequence of Monetary Items: Fisher-Weil and Redington's Theorems

L'immunizzazione finanziaria di successioni di poste monetarie: una estensione dei teoremi di Fisher-Weil e di Redington

Luigi Romano and Donato Scolozzi

Abstract In this paper we study the financial immunization when the cash flows are represented by a sequence of real numbers. Old Fisher-Weil's and Redington's results are generalized in this setting.

Abstract *In questo lavoro si studia il problema della immunizzazione finanziaria semi deterministica per successioni di poste monetarie. Si considerano due strutture di tassi di interesse: una per le poste attive ed una per le poste passive e in queste condizioni vengono estesi i classici risultati di Fisher-Weil e di Redington.*

Key words: immunization, cash flows, duration

1 Introduction

The Theory of Financial Immunization examines the problem of the interest rate risk and studies the conditions in which it is possible to manage asset cash flows by means of liability cash flows. In 1952 Redington, during a communication held at the college of Actuarial Sciences in London, introduced a model to manage liability cash flows. This result takes into account the initial financial structure, which is represented by the instantaneous intensity interest $\delta(t,s)$; the interest changes (shift) are constant and represented by the real number z which is added to $\delta(t,s)$ determining the new financial structure through the new instantaneous intensity $\delta(t,s)+z$. The same setting is used by [9] in their 1971 work, where they examine the problem of the management of one single liability through an asset cash flow. From this setting, it follows that, if the assigned function $\delta(t,s)$ is integrable with respect to the second variable s, the relative value function $v(t,s)$ and relative interest rate function $i(t,s)$ used in the contract examined, are continuous with respect to the same variable. In fact, they are connected with the instantaneous intensity by the following relations:

$$v(t,s) = e^{-\int_t^s \delta(t,u)du}, \quad i(t,s) = e^{\frac{1}{s-t}\int_t^s \delta(t,u)du} - 1.$$

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This model examines the case with constant interest rate, but not the case with discontinuous interest rate with respect to s. The proof of their immunization theorem use the Macaulay duration concept, 1938. All these results are exposed in [8].

F. Gozzi [10] examines a unified and compact approach to the semi-deterministic financial immunization theory. Another consideration concerns the sign of the instantaneous intensity of interest which, as known, must be nonnegative; so the shifts z are admissible if also the new instantaneous intensity of interest is nonnegative. A case with admissible and constant shift for contracts with finite cash flows is examined in [5, 6], together with the case in which all cash flow average durations coincide. In a later work, [3] examine contracts with finite cash flows, in which the value function is the market's variable base, and it is accepted that the value function must be "'stationary".

The present paper does not suppose the existence of the function $\delta(t,s)$ and it assumes the value function $v(t,s)$ as the market's variable base, so that it is possible to include settings with discontinuous interest rates. Besides, this work examines a sequence of asset cash flows and liability cash flows, whose sign is not necessarily constant. Finally, it examines constant shifts of additive and admissible type. Redington [12] and Fisher Weil's [9] theorems are generalized in this setting.

Other authors, mentioned in references, have already examined the case with variable shift, which, from this point of view, will be examined in a later work.

2 The actual value function and the average duration.

Defining a function $v : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, we have a value function if, and only if, the following three properties are verified:

1. $v(t,s) > 0 \quad \forall (t,s) \in \mathbb{R} \times \mathbb{R} \quad \text{with} \quad t \leq s$;
2. $v(t,t) = 1 \quad \forall (t,t) \in \mathbb{R} \times \mathbb{R}$;
3. $v(t,s_1) \geq v(t,s_2) \quad \forall (t,s_1), (t,s_2) \in \mathbb{R} \times \mathbb{R} \quad \text{with} \quad s_1 \leq s_2$.

The simple and compound capitalization regimes are very important in finance. The relative value functions are respectively:

$$v(t,s) = \frac{1}{1+a(s-t)} \quad \forall (t,s) \in \mathbb{R} \times \mathbb{R} \quad \text{with} \quad s > t - \frac{1}{a}$$

and

$$v(t,s) = e^{-a(s-t)} \quad \forall (t,s) \in \mathbb{R} \times \mathbb{R}$$

where a, in both cases, is the interest rate on one period.

Also, for example, functions of the type:

$$v(t,s) = \frac{1}{1+a(s-t)^{\alpha}} \quad \forall (t,s) \in \mathbb{R} \times \mathbb{R} \quad \text{with} \quad s \geq t$$

with $a > 0$ and $\alpha > 0$, are value functions; they decrease of order α.

It must be noted that if, as in the examples before, the value function has partial derivative with respect to variable s, we can define, as we know, the instantaneous intensity interest function $\delta(t,s)$ in the following way:

$$\delta(t,s) = -\frac{1}{v(t,s)} \frac{\partial v}{\partial s}(t,s)$$

The issue date t, a sequence of maturities $\mathbf{t} = (t_j)_{j\in\mathbb{N}}$, with $t < t_1 \leq t_j < t_{j+1} \quad \forall j \in \mathbb{N}$, and a sequence of cash flows $\mathbf{x} = (x_j)_{j\in\mathbb{N}}$ with $x_j \in \mathbb{R} \quad \forall j \in \mathbb{N}$, be assigned such that each single-payment x_j is available at time t_j.

As we known, we define the actual value at time t of cash flow $\mathbf{x}$, with respect to the assigned value function v, the following summation:

$$W(t,\mathbf{x}) = \sum_{j=1}^{+\infty} x_j v(t,t_j)$$

assuming that the series converges.

In financial terms, it is possible to define the real number $W(t,\mathbf{x})$ as the price to pay at time t, to buy the right to get, at every maturity t_j, the single-payment x_j.

The following results provide the conditions sufficient to define the price as a real number.

We'll examine cases in absence of interest rate shift, and then we'll take into consideration a few situations in presence of interest rate shift.

The following result considers the case of value function with decrease polynomial.

Theorem 1. *Let us suppose that:*

1. $\exists\, a > 0, \alpha > 1, c \geq 1 : \ 0 < v(t,s) \leq \frac{c}{1+a(s-t)^\alpha}$ $\forall t,s \in \mathbb{R}$ *with* $s \geq t$
2. $\exists\, \tau > 0 \ : t_j - t \geq \tau j \quad \forall j \in \mathbb{N}$
3. $\exists\, r \in [0,+\infty[,\ \beta \in [0,\alpha-1[: \ |x_j| \leq r\,(t_j - t)^\beta$

Thesis:

$$|W(t,\boldsymbol{x})| \leq \sum_{j=1}^{+\infty} |x_j|\, v(t,t_j) \leq \frac{rc}{a\tau^\alpha} \sum_{j=1}^{+\infty} \frac{1}{j^{\alpha-\beta}}$$

Proof. $\forall n \in \mathbb{N}$ we have:

$$\left| \sum_{j=1}^{n} x_j v(t,t_j) \right| \leq \sum_{j=1}^{n} |x_j|\, v(t,t_j) \leq \sum_{j=1}^{n} \frac{cr\,(t_j-t)^\beta}{1+a(t_j-t)^\alpha} \leq$$

$$\frac{cr}{a} \sum_{j=1}^{n} \frac{1}{(t_j-t)^{\alpha-\beta}} \leq \frac{rc}{a\tau^\alpha} \sum_{j=1}^{n} \frac{1}{j^{\alpha-\beta}} \leq \frac{rc}{a\tau^\alpha} \sum_{j=1}^{+\infty} \frac{1}{j^{\alpha-\beta}}$$

Since the term on the right represents the sum of the harmonic series with exponent $\alpha - \beta > 1$, we can deduce that the assigned series converges absolutely. $\diamondsuit$

The following result examines the case of value function with exponential decrease.

Theorem 2. *Let us suppose that:*

1. $\exists m > 0 \ : \ 0 < v(t,s) \leq e^{-m(s-t)}$
2. $\exists\, \tau > 0 \ : t_j - t \geq \tau j \quad \forall j \in \mathbb{N}$
3. $\exists r \in [0,m[\ : \ |x_j| \leq e^{r(t_j-t)}$

Thesis:

$$|W(t,\boldsymbol{x})| \leq \sum_{j=1}^{+\infty} |x_j|\, v(t,t_j) \leq \frac{e^{\tau(r-m)}}{1-e^{\tau(r-m)}}$$

Proof. $\forall n \in \mathbb{N}$ we have:

$$\left| \sum_{j=1}^{n} x_j v(t,t_j) \right| \leq \sum_{j=1}^{n} |x_j|\, v(t,t_j) \leq \sum_{j=1}^{n} e^{(r-m)(t_j-t)} \leq$$

$$\sum_{j=1}^{n}\left[e^{\tau(r-m)}\right]^{j}=e^{\tau(r-m)}\frac{1-e^{\tau n(r-m)}}{1-e^{\tau(r-m)}}\leq\frac{e^{\tau(r-m)}}{1-e^{\tau(r-m)}}$$

If we reckon the limit for $n\to+\infty$ we have the thesis. $\diamondsuit$

In the following case, the interest rate shift will be examined.

In the classic theory of the semi-deterministic financial immunization, the instantaneous intensity of interest function $\delta(t,s)$ is assumed as basic variable, for it we have $\delta(t,s)\geq 0, \quad \forall t\leq s$. In the further assumption that this is, according to Lebesgue, locally integrable with respect to variable s, the relative value function $v(t,s)$ is defined by $v(t,s)=e^{-\int_t^s \delta(t,u)du}$ and it is absolutely continuous with respect to s. It follows that also the interest rate function defined as $i(t,s)=\left[\frac{1}{v(t,s)}\right]^{\frac{1}{s-t}}-1$, is continuous with respect to variable s.

The interest rate shifts which are considered in the classic theory, are determined by a constant quantity z which is added to the function $\delta(t,s)$, so that we have the function $\gamma(t,s)=\delta(t,s)+z$, and for this reason z is called additive shift. Obviously we also want that $\gamma(t,s)=\delta(t,s)+z\geq 0, \quad \forall t\leq s$. In this case, in the classic theory, z is defined as an admissible shift.

Obviously also $\gamma(t,s)$, which is integrable like $\delta(t,s)$, defines the value function $v(t,s)e^{-z(s-t)}$ which is absolutely continuous with respect to s.

In the present work we are interested to examine contracts which are subscribed assigning the value function. In this way it is possible to include into analysis also cases, in which, the instantaneous intensity of interest function is not defined, because the value function itself may not be continuous with respect to variable s, in analogy with the correspondent interest rates.

In that case, we assume the value function as the basic variable. So, if z is a constant shift that changes the rates to save its additivity nature, we must consider, in analogy with the classic case, the new value function $v(t,s)e^{-z(s-t)}$. Finally, we can say that z is admissible if the new function is monotonically decreasing with respect to s. This is verified if $z\geq 0$.

The following definition synthesizes what we have just stated.

Let $v(t,s)$ be a value function and $z\in\mathbb{R}$, we can say that z is an ”admissible additive shift” if, and only if, the function $v(t,s)e^{-z(s-t)}$ is still a value function.

So if $z\geq 0$, then it is admissible.

In the previous case, and in presence of a z shift constant and admissible additive, by means of the new value function $v(t,s)e^{-z(s-t)}$, it is possible to determine, by addition, the new actual value function:

$$W(t,\mathbf{x},z)=\sum_{j=1}^{+\infty}x_j v(t,t_j)e^{-z(t_j-t)}$$

Theorem 3. *Let* $W(t,\boldsymbol{x})=\sum_{j=1}^{+\infty}x_j v(t,t_j)$ *be absolutely convergent.*

Thesis:

$$|W(t,\boldsymbol{x},z)|\leq\sum_{j=1}^{+\infty}\left|x_j\right|v(t,t_j)e^{-z(t_j-t)}\leq\sum_{j=1}^{+\infty}\left|x_j\right|v(t,t_j)\quad \forall z\in[0,+\infty[.$$

Besides, the following function is continuous:

$$W(t,\boldsymbol{x},\cdot):z\in[0,+\infty[\longrightarrow W(t,\boldsymbol{x},z)=\sum_{j=1}^{+\infty}x_j v(t,t_j)e^{-z(t_j-t)}\in\mathbb{R}$$

.

Proof. It is sufficient to observe that the increases indicated in the thesis are obvious, and as it is assumed that the series $\sum_{j=1}^{+\infty} x_j v(t,t_j)$ converges absolutely, the series that defines the actual value function converges completely, and so uniformly, with respect to $z \in [0,+\infty[$. As the addend functions that make the series $W(t,\mathbf{x},z)$ are continuous, the thesis follows. $\diamondsuit$

The following result provides a condition to the existence of the actual value function's derivatives.

Theorem 4. *Let us suppose that $h \in \mathbb{N}$, such that the series*

$$\sum_{j=1}^{+\infty} (t_j - t)^h x_j v(t,t_j)$$

is absolutely convergent.

Thesis: $\forall\ k = 1,2,\ldots,h$ *e* $\forall\ z \in [0,+\infty[$ *we have:*

$$\exists \quad \frac{\partial^k W}{\partial z^k}(t,\mathbf{x},z) = (-1)^k \sum_{j=1}^{+\infty} (t_j - t)^k x_j v(t,t_j) e^{-z(t_j - t)}$$

Proof. From the hypothesis given, we can deduce that the actual value function is well defined. It is sufficient to observe that, $\forall\ n \in \mathbb{N}$, results in:

$$\sum_{j=1}^{n} |x_j| \, v(t,t_j) \leq \frac{1}{(t_1 - t)^h} \sum_{j=1}^{n} (t_j - t)^h |x_j| \, v(t,t_j)$$

thus, also the series $W(t,\mathbf{x}) = \sum_{j=1}^{+\infty} x_j v(t,t_j)$ is absolutely convergent.

Moreover, the same function is derivable until order h. To have this result, first of all, it is sufficient to verify that, assigning $k = 1,2,\ldots,h$ the following series:

$$\sum_{j=1}^{+\infty} (t_j - t)^k x_j v(t,t_j).$$

is absolutely convergent.

Considering $n \in \mathbb{N}$ we have:

$$\left| \sum_{j=1}^{n} (t_j - t)^k x_j v(t,t_j) \right| \leq \sum_{j=1}^{n} (t_j - t)^k |x_j| \, v(t,t_j) =$$

$$\sum_{j=1}^{n} \frac{1}{(t_j - t)^{h-k}} (t_j - t)^h |x_j| \, v(t,t_j) \leq \sum_{j=1}^{n} \frac{1}{(t_1 - t)^{h-k}} (t_j - t)^h |x_j| \, v(t,t_j) =$$

$$\frac{1}{(t_1 - t)^{h-k}} \sum_{j=1}^{n} (t_j - t)^h |x_j| \, v(t,t_j)$$

Reckoning the limit for $n \to +\infty$ we have the result.

To complete the proof about the existence of the derivatives until order h, we use induction reasoning. Be $k = 1$.

According to what we have stated above, the following series

$$-\sum_{j=1}^{+\infty}(t_j-t)\,x_j v(t,t_j)$$

converges absolutely.

Besides, being $\forall\ n\in\mathbb{N}$:

$$\sum_{j=1}^{n}(t_j-t)\left|x_j\right|v(t,t_j)e^{-z(t_j-t)}\leq\sum_{j=1}^{n}(t_j-t)\left|x_j\right|v(t,t_j)\quad\forall z\in[0,+\infty[$$

the series that determines the actual value function converges uniformly. It allows, $\forall z\in[0,+\infty[$, the following relation :

$$\exists\quad\frac{\partial}{\partial z}\left(\sum_{j=1}^{n}x_j v(t,t_j)e^{-z(t_j-t)}\right)=-\sum_{j=1}^{n}(t_j-t)x_j v(t,t_j)e^{-z(t_j-t)}$$

Similarly, for $k=1,2,\ldots,h$, it is possible to prove that:

$$\exists\quad\frac{\partial^k}{\partial z^k}\left(\sum_{j=1}^{n}x_j v(t,t_j)e^{-z(t_j-t)}\right)=(-1)^k\sum_{j=1}^{n}(t_j-t)^k x_j v(t,t_j)e^{-z(t_j-t)}.\diamondsuit$$

The following results summarize two situations very important in finance.

The following result examines the decreasing polynomial value function.

Theorem 5. *Let us suppose that:*

1. $\exists\ a>0,\alpha>1,c\geq1:\ 0<v(t,s)\leq\frac{c}{1+a(s-t)^\alpha}\ \forall t,s\in\mathbb{R}$ *with* $s\geq t$
2. $\exists\ \tau>0\ \ :\ t_j-t\geq\tau j\quad\forall j\in\mathbb{N}$
3. $\exists\ r\in[0,+\infty[,\ \beta\in[0,\alpha-1[:\ \ \left|x_j\right|\leq r\,(t_j-t)^\beta$

Thesis:

$$\forall z\in[0,+\infty[\quad|W(t,\boldsymbol{x},z)|\leq\sum_{j=1}^{+\infty}\left|x_j\right|v(t,t_j)e^{-z(t_j-t)}\leq\frac{rc}{a\tau^\alpha}\sum_{j=1}^{+\infty}\frac{e^{-z\tau j}}{j^{\alpha-\beta}}$$

It is easy to observe that, if the trend of the value function is polynomial type, the actual value function may not be defined as $z<0$.

The following theorem examines some conditions that ensure the existence of derivatives.

Theorem 6. *Let us suppose that:*

1. $\exists\ a>0,\alpha>1,c\geq1:\ 0<v(t,s)\leq\frac{c}{1+a(s-t)^\alpha}\ \forall t,s\in\mathbb{R}$ *with* $s\geq t$
2. $\exists\ \tau>0,\ \eta>0$ *with* $\tau\leq\eta\ \ :\tau j\leq t_j-t\leq\eta j\quad\forall j\in\mathbb{N}$;
3. $\exists\ r\in[0,+\infty[,\ \beta\in[0,\alpha-1[:\ \ \left|x_j\right|\leq r\,(t_j-t)^\beta$

Thesis: $\quad\forall k=1,2,\ldots,h;\ \ \forall z\in[0,+\infty[$

$$\exists\frac{\partial^k W}{\partial z^k}(t,\boldsymbol{x},z)=(-1)^k\sum_{j=1}^{+\infty}(t_j-t)^k x_j v(t,t_j)e^{-z(t_j-t)}$$

and we have:

$$\left|\frac{\partial^k W}{\partial z^k}(t,\boldsymbol{x},z)\right|\leq\frac{\eta rc}{a\tau^\alpha}\sum_{j=1}^{+\infty}\frac{e^{-z\tau j}}{j^{\alpha-\beta-k}}.$$

The following result examines the case of value function with exponential decrease. It is an example where the actual value function is defined and derivable also for appropriates $z < 0$.

Theorem 7. *Let us suppose that:*

1. $\exists m > 0 \;\; : \;\; 0 < v(t,s) \leq e^{-m(s-t)}$
2. $\exists\, \tau > 0 \;\; : t_j \geq t + \tau j \quad \forall j \in \mathbb{N}$
3. $\exists r \in]0,m[\;\; : \;\; \left| x_j \right| \leq e^{r(t_j - t)}$

Thesis:

$$\forall z \in]r-m,+\infty[\quad |W(t,\boldsymbol{x},z)| = \left| \sum_{j=1}^{+\infty} x_j v(t,t_j) e^{-z(t_j-t)} \right| \leq \frac{e^{\tau(r-m-z)}}{1-e^{\tau(r-m-z)}}$$

Regarding the existence of the derivates we can consider the following theorem that ensures the existence of the derivatives of every order.

Theorem 8. *Let us suppose that:*

1. $\exists m > 0 \;\; : \;\; 0 < v(t,s) \leq e^{-m(s-t)}$
2. $\exists\, \tau > 0,\ \exists\ \eta > 0$ *with* $\tau \leq \eta \;\; : \tau j \leq t_j - t \leq \eta j \quad \forall j \in \mathbb{N}$
3. $\exists r \in]0,m[\;\; : \;\; \left| x_j \right| \leq e^{r(t_j - t)}$

Thesis: fixed $k \in \mathbb{N}$, $\forall\, z \in]r-m,+\infty[$ *we have:*

$$\exists \frac{\partial^k W}{\partial z^k}(t,\boldsymbol{x},z) = (-1)^k \sum_{j=1}^{+\infty} (t_j - t)^k x_j v(t,t_j) e^{-z(t_j-t)}$$

The above mentioned theorems include constant cash flows (perpetual income) and the capitalization laws of compound and polynomial increase. The law of simple capitalization has not been considered in the previous results, if we want to keep the hypothesis that monetary items may be constant.

Now we shall examine some financial indexes very significant in the financial analysis.

In the conditions initially established, if the series $\sum_{j=1}^{+\infty}(t_j-t)x_j v(t,t_j)$ converges absolutely and if we have $\sum_{j=1}^{+\infty} x_j v(t,t_j) \neq 0$, we define "financial average duration" of order 1, at time t, of the cash flows $\mathbf{x}$, with respect to the assigned value function $v(t,s)$, the following fraction:

$$D(t,\mathbf{x}) = \frac{\sum_{j=1}^{+\infty}(t_j-t)x_j v(t,t_j)}{\sum_{j=1}^{+\infty} x_j v(t,t_j)}$$

Theorem 9. *Let us suppose that the series* $\sum_{j=1}^{+\infty}(t_j-t)x_j v(t,t_j)$ *is absolutely convergent, and be also* $W(t,\boldsymbol{x},0) \neq 0$.

Thesis:

$$\exists \quad D(t,\boldsymbol{x}) = \left| \frac{1}{W(t,\boldsymbol{x},0)} \frac{\partial W}{\partial z}(t,\boldsymbol{x},0) \right| = \left| \frac{\partial \log |W|}{\partial z}(t,\boldsymbol{x},0) \right|.$$

Proof. In the previous hypotheses, the actual value function $W(t,\mathbf{x},z)$ is derivable of order 1 with respect to z. Being $W(t,\mathbf{x},0) \neq 0$, we have the thesis. $\diamondsuit$

Let $k \in \mathbb{N}$ be assigned in the conditions initially established, if the series $\sum_{j=1}^{+\infty}(t_j-t)^k x_j v(t,t_j)$ converges absolutely and if is $\sum_{j=1}^{+\infty} x_j v(t,t_j) \neq 0$, we define "financial average duration", of order k in t, of the cash flow of single-payment $\mathbf{x}$ with respect to the assigned value function $v(t,s)$, the following fraction:

$$D^{(k)}(t,\mathbf{x}) = \frac{\sum_{j=1}^{+\infty}(t_j-t)^k x_j v(t,t_j)}{\sum_{j=1}^{+\infty} x_j v(t,t_j)}$$

If the average duration of order k exists, let then the average durations of order $\ell \leq k$ also exist.

Theorem 10. *Let us suppose that the series* $\sum_{j=1}^{+\infty}(t_j-t)^k x_j v(t,t_j)$ *is absolutely convergent and be also that* $W(t,\boldsymbol{x},0) \neq 0$.

Thesis:

$$\exists \quad D^{(k)}(t,\boldsymbol{x}) = \left| \frac{1}{W(t,\boldsymbol{x},0)} \frac{\partial^k W}{\partial z^k}(t,\boldsymbol{x},0) \right|.$$

Proof. In the given hypotheses, the actual value function $W(t,\mathbf{x},z)$ is derivable until order k, with respect to z, and we have:

$$\frac{\partial^k W}{\partial z^k}(t,\mathbf{x},0) = (-1)^{(k)} \sum_{j=1}^{+\infty}(t_j-t)^k x_j v(t,t_j)$$

.

Being $W(t,\mathbf{x},0) \neq 0$, we have the thesis. $\diamondsuit$

The following properties are valid.

Theorem 11. *If* $D^{(k)}(t,\boldsymbol{x})$ *exists, then:*

1. *if* $x_j \geq 0 \quad \forall j \in \mathbb{N}$ *it results:* $D^{(k)}(t,\boldsymbol{x}) \geq (t_1-t)^k$
2. $D^{(k)}(t,\lambda\boldsymbol{x}) = D^{(k)}(t,\boldsymbol{x}) \quad \forall \lambda \in \mathbb{R}, \quad \lambda \neq 0.$

Only note that, if x is an asset cash flow and if we have $\lambda > 0$, then also λx is still an asset cash flow; instead, if we have $\lambda < 0$, then the cash flow λx becomes a liability cash flow, but the average duration is the same.

3 The immunization with respect to constant shift

The classic theory examines asset and liability cash flows in a finite quantity and it hypothesizes the existence of the instantaneous intensity of interest function $\delta(t,s)$. This hypothesis, as already observed, imposes the value function (and also the interest rates) to admit partial derivative, of order 1, with respect to the maturity s and therefore to be continuous. If the variations of the interest rate examined, represented by shift z, are summed to $\delta(t,s)$, we have the new instantaneous intensity of interest function $\delta(t,s)+z$; for this reason the shift is called additive.

This paper examines a generalization of financial immunization classic theory to the sequence of asset and liability cash flows. In addition, it examines the value function and it does not hypothesize the existence of the instantaneous intensity of interest function. In this way it also includes the study of the interest rate structures which might be discontinuous.

The following conditions have been hypothesized:

a) an issue date t, a sequence of maturities $\mathbf{t} = (t_j)_{j\in\mathbb{N}}$, with $t < t_1 \leq t_j < t_{j+1} \quad \forall j \in \mathbb{N}$;
b) a sequence of monetary items $\mathbf{x} = (x_j)_{j\in\mathbb{N}}$ with $x_j \in \mathbb{R} \quad \forall j \in \mathbb{N}$, such that every x_j is available at time t_j. This cash flow will be denoted as "assets";
c) a sequence of monetary items $\mathbf{y} = (y_j)_{j\in\mathbb{N}}$ with $y_j \in \mathbb{R} \quad \forall j \in \mathbb{N}$, such that every y_j is a payment due at time t_j. This cash flow will be denoted as "liabilities";
d) a value function $v(t,s)$ that discounts the assets, and another value function $w(t,s)$ that discounts the liabilities.

This work examines situation when we want to immunize the liability cash flow $\mathbf{y}$ with the asset cash flow $\mathbf{x}$.

The cash flow $\mathbf{x}$ immunizes cash flow $\mathbf{y}$ if, in the hypothesis that the series $\sum_{j=1}^{+\infty} x_j v(t,t_j)$ and $\sum_{j=1}^{+\infty} y_j w(t,t_j)$ are regular, we have:

$$\sum_{j=1}^{+\infty} x_j v(t,t_j) \geq \sum_{j=1}^{+\infty} y_j w(t,t_j)$$

The following result considers the problem of the coverage of one single liability L at time H.

Theorem 12 (FISHER-WEIL). *Let an issue date t, a sequence of maturities $\mathbf{t} = (t_j)_{j\in\mathbb{N}}$, with $t < t_1$, $t_j < t_{j+1} \quad \forall j \in \mathbb{N}$, and a sequence of monetary items $\boldsymbol{x} = (x_j)_{j\in\mathbb{N}}$ with $x_j \in \mathbb{R} \quad \forall j \in \mathbb{N}$, be assigned, every item payable at time t_j and be $v(t,s)$ the relative value function. Let also $W(t,\boldsymbol{x}) = \sum_{j=1}^{+\infty} x_j v(t,t_j)$ be the price, at time t, of cash flow $\boldsymbol{x}$.*

A single payment L be assigned at time H, let $w(t,s)$ the relative value function and $W(t,L) = Lw(t,H)$. The following hypotheses be verified:

1. *$x_j \geq 0 \ \ \forall j \in \mathbb{N}$*
2. *$W(t,\boldsymbol{x}) = W(t,L)$;*
3. *$\exists D^{(2)}(t,\boldsymbol{x})$;*
4. *let $z \in \mathbb{R}$ be a shift such that the functions $v(t,s)e^{-z(s-t)}$ and $w(t,s)e^{-z(s-t)}$ are still value functions.*

Thesis:

$$(W(t,\boldsymbol{x},z) \geq W(t,L,z) \ \ \forall\, z \ \ \text{"admissible"}) \Longleftrightarrow D(t,\boldsymbol{x}) = H - t$$

Proof. If $W(t,L,z) \neq 0$, let us consider the function

$$Q(z) = \frac{W(t,\mathbf{x},z)}{W(t,L,z)} = \frac{1}{W(t,L)} \sum_{j=1}^{+\infty} x_j v(t,t_j) e^{-z(H-t_j)}$$

We have the result $Q(z) \geq 1, \quad \forall z$ admissible. It is sufficient proof that $z = 0$ is the minimum point for Q.

With reference to this, we must observe that because the average duration of order 2 exists, so to the function Q has derivatives of order 1 and 2:

$$Q'(z) = \frac{1}{W(t,L)} \sum_{j=1}^{+\infty} (t_j - H) x_j v(t,t_j) e^{-z(H-t_j)}$$

$$Q''(z) = \frac{1}{W(t,L)} \sum_{j=1}^{+\infty} (t_j - H)^2 x_j v(t,t_j) e^{-z(H-t_j)}$$

Therefore the function Q is strictly convex and $z = 0$ is the minimum point if, and only if, we have:

$$Q'(0) = 0.$$

It is simple to verify that:

$$Q'(0) = D(t,\mathbf{x}) - H + t$$

from which we have the thesis.$\diamondsuit$

Let an issue date t be assigned, together with a sequence of maturities $\mathbf{t} = (t_j)_{j\in\mathbb{N}}$, with $t < t_1$, $t_j < t_{j+1} \quad \forall j \in \mathbb{N}$, a sequence of monetary items $\mathbf{x} = (x_j)_{j\in\mathbb{N}}$ with $x_j \in \mathbb{R} \quad \forall j \in \mathbb{N}$, each of them payable at time t_j and let $v(t,s)$ be the relative value function.

Let H be a date, and let us consider the income at time H of cash flow $\mathbf{x}$, determined by the reinvestment of the items x_j, with $t_j \le H$, from t_j to H, and the discounting of the items x_j payable at time t_j with $t_j > H$.

Thus the formula:

$$R(H,\mathbf{x}) = \sum_{t_j \le H} \frac{x_j}{v(t_j,H)} + \sum_{t_j > H} x_j v(H,t_j)$$

if the series $\sum_{t_j>H} x_j v(H,t_j)$ is regular.

We want to verify whether a time $\overline{H}$ exists, such that, if the interest rates change because there is an admissible additive shift z, if

$$R(\overline{H},\mathbf{x},z) = \sum_{t_j \le H} \frac{x_j}{v(t_j,H)e^{-z(H-t_j)}} + \sum_{t_j > H} x_j v(H,t_j) e^{-z(t_j-H)}$$

is the relative income, we have that:

$$R(\overline{H},\mathbf{x},z) \ge R(\overline{H},\mathbf{x},0).$$

Unfortunately this problem is difficult to solve because at time t the financial functions are random. In order to carry out an initial analysis, it would be appropriate to simplify the problem.

Theorem 13. *[Reduced income] Let $v(t,s)$ be the value function such that we have:*

1. $v(t,H) = v(t,t_j)v(t_j,H) \;\; \forall t_j \;\; with \;\; t \le t_j \le H$
2. $v(t,t_j) = v(t,H)v(H,t_j) \;\; \forall t_j \;\; with \;\; t \le H \le t_j$

Thesis: $R(H,\boldsymbol{x}) = \frac{1}{v(t,H)} W(t,\boldsymbol{x})$.

Theorem 14 (Existence of Optimal Time for Disinvestment.). *Let $v(t,s)$ be the value function, such that we have:*

1. $v(t,H) = v(t,t_j)v(t_j,H) \;\; \forall t_j \;\; with \;\; t \le t_j \le H$
2. $v(t,t_j) = v(t,H)v(H,t_j) \;\; \forall t_j \;\; with \;\; t \le H \le t_j$
3. $x_j \ge 0 \;\; \forall j \in \mathbb{N}$
4. $\exists D^{(2)}(t,\boldsymbol{x})$.

Besides, let $z \in \mathbb{R}$ be a shift such that $v(t,s)e^{-z(s-t)}$ is still a value function.

Thesis: $H = t + D(t,\boldsymbol{x})$ is the best time for disinvestment.

Therefore, for all z admissible, the result is:

$$R(t + D(t,\boldsymbol{x}),\boldsymbol{x}) \le R(t + D(t,\boldsymbol{x}),\boldsymbol{x},z).$$

Proof. The income function, in the previous hypotheses and if, we have a shift z, is:

$$R(H,\mathbf{x},z) = \frac{1}{v(t,H)} \sum_{j=1}^{+\infty} x_j v(t,t_j) e^{-z(H-t_j)}.$$

From the hypothesis of existence of the average duration of order 2, we can deduce the existence of the derivatives, of order 1 and 2, with respect to z and we have:

$$R'(H,\mathbf{x},z) = \frac{1}{v(t,H)} \sum_{j=1}^{+\infty} (t_j - H) x_j v(t,t_j) e^{-z(H-t_j)}$$

$$R''(H,\mathbf{x},z) = \frac{1}{v(t,H)} \sum_{j=1}^{+\infty} (H - t_j)^2 x_j v(t,t_j) e^{-z(H-t_j)}$$

Therefore the function is strictly convex and $z = 0$ is the minimum point if, and only if, $H = t + D(t,\mathbf{x})$ which represents the thesis. $\diamondsuit$

The following result provides a condition necessary to manage the liability with the asset by using only the average duration of order 1.

Theorem 15. *Let an issue date t, a sequence of maturities $\boldsymbol{t} = (t_j)_{j\in\mathbb{N}}$, with $t < t_1$, $t_j < t_{j+1} \quad \forall j \in \mathbb{N}$, a sequence of monetary items $\boldsymbol{x} = (x_j)_{j\in\mathbb{N}}$ with $x_j \in \mathbb{R} \quad \forall j \in \mathbb{N}$, be assigned, every item payable at time t_j; let $v(t,s)$ be the value function and $W(t,\boldsymbol{x}) = \sum_{j=1}^{+\infty} x_j v(t,t_j)$.*

Let a sequence of monetary items $\boldsymbol{y} = (y_j)_{j\in\mathbb{N}}$ with $y_j \in \mathbb{R} \quad \forall j \in \mathbb{N}$ be also fixed, each of which represents a payment due at time t_j. Let $w(t,s)$ be the value discount function and $W(t,\boldsymbol{y}) = \sum_{j=1}^{+\infty} y_j w(t,t_j)$.

The following hypotheses be verified:

1. $\exists W(t,\boldsymbol{x})$, $\exists W(t,\boldsymbol{y})$ *and* $\quad W(t,\boldsymbol{x}) = W(t,\boldsymbol{y})$;
2. $\exists D(t,\boldsymbol{x})$, $\exists D(t,\boldsymbol{y})$ *and* $\quad D(t,\boldsymbol{x}) \neq D(t,\boldsymbol{y})$;
3. *let z be an admissible additive shift;*

Thesis:

I) $D(t,\boldsymbol{x}) > D(t,\boldsymbol{y}) \implies \exists a > 0$ *such that:*

$$\forall z \in [-a,0[, \ \text{admissible} \ W(t,\boldsymbol{x},z) > W(t,\boldsymbol{y},z)$$

$$\forall z \in]0,a], \ \text{admissible} \ W(t,\boldsymbol{x},z) < W(t,\boldsymbol{y},z)$$

II) $D(t,\boldsymbol{x}) < D(t,\boldsymbol{y}) \implies \exists a > 0$ *such that:*

$$\forall z \in [-a,0[, \ \text{admissible} \ W(t,\boldsymbol{x},z) < W(t,\boldsymbol{y},z)$$

$$\forall z \in]0,a], \ \text{admissible} \ W(t,\boldsymbol{x},z) > W(t,\boldsymbol{y},z)$$

Proof. Let us consider the net actual value function:

$$W(t,\mathbf{x}-\mathbf{y},z) = \sum_{j=1}^{+\infty} [x_j v(t,t_j) - y_j w(t,t_j)]\, e^{-z(t_j-t)}.$$

We have that $W(t,\mathbf{x}-\mathbf{y},z)=0$, continuous for all z admissible, derivable with respect to z.
In $z=0$ so we have:

$$\frac{\partial W}{\partial z}(t,\mathbf{x}-\mathbf{y},0)=W(t,\mathbf{y},0)D(t,\mathbf{y})-W(t,\mathbf{x},0)D(t,\mathbf{x})$$

from which it is simple to deduce the thesis. $\diamondsuit$

The following result provides a sufficient condition to obtain immunization should the average durations of order 1 coincide: $D(t,\mathbf{x})=D(t,\mathbf{y})$.

Theorem 16 (di REDINGTON). *Let an issue date t, a sequence of maturities $\boldsymbol{t}=(t_j)_{j\in\mathbb{N}}$, with $t<t_1$, $t_j<t_{j+1}\quad\forall j\in\mathbb{N}$, and a sequence of monetary items $\boldsymbol{x}=(x_j)_{j\in\mathbb{N}}$ with $x_j\in\mathbb{R}\quad\forall j\in\mathbb{N}$ be assigned, every item payable at time t_j. Let $v(t,s)$ be the value discount function, and $W(t,\boldsymbol{x})=\sum_{j=1}^{+\infty}x_j v(t,t_j)$.*

Also a sequence of monetary items $\boldsymbol{y}=(y_j)_{j\in\mathbb{N}}$ with $y_j\in\mathbb{R}\quad\forall j\in\mathbb{N}$ be assigned, each of which is a single-payment due at time t_j. Let $w(t,s)$ be the value function and $W(t,\boldsymbol{y})=\sum_{j=1}^{+\infty}y_j w(t,t_j)$.

Let the following hypotheses be verified:

1. *$\exists W(t,\boldsymbol{x})$, $\exists W(t,\boldsymbol{y})$ is $\quad W(t,\boldsymbol{x})=W(t,\boldsymbol{y})$;*
2. *$\exists D(t,\boldsymbol{x})$, $\exists D(t,\boldsymbol{y})$ is $\quad D(t,\boldsymbol{x})=D(t,\boldsymbol{y})$;*
3. *$\exists D^{(2)}(t,\boldsymbol{x})$, $\exists D^{(2)}(t,\boldsymbol{y})$ is $\quad D^{(2)}(t,\boldsymbol{x})>D^{(2)}(t,\boldsymbol{y})$;*
4. *let z be an admissible additive shift;*

Thesis:

$$\exists a>0:\quad \forall z\in[-a,a],\ \text{admissible, it results in:}\quad W(t,\boldsymbol{x},z)\geq W(t,\boldsymbol{y},z).$$

Proof. The net actual value function needs to be considered:

$$W(t,\mathbf{x}-\mathbf{y},z)=\sum_{j=1}^{+\infty}[x_j v(t,t_j)-y_j w(t,t_j)]\,e^{-z(t_j-t)}.$$

For the previous hypotheses the result is: $W(t,\mathbf{x}-\mathbf{y},z)=0$.
In addition, it has derivables of order 1 and 2 for all z admissible:

$$\frac{\partial W}{\partial z}(t,\mathbf{x}-\mathbf{y},0)=W(t,\mathbf{y},0)D(t,\mathbf{y})-W(t,\mathbf{x},0)D(t,\mathbf{x})=0$$

$$\frac{\partial^2 W}{\partial z^2}(t,\mathbf{x}-\mathbf{y},0)=W(t,\mathbf{x},0)D^{(2)}(t,\mathbf{x})-W(t,\mathbf{y},0)D^{(2)}(t,\mathbf{y})>0$$

from this, it is simple to have the thesis. $\diamondsuit$

The following result is an example of loss.

Theorem 17. *Let us consider: an issue date t, a sequence of maturities*
$\boldsymbol{t}=(t_j)_{j\in\mathbb{N}}$, with $t<t_1$, $t_j<t_{j+1}\quad\forall j\in\mathbb{N}$, and a sequence of monetary items $\boldsymbol{x}=(x_j)_{j\in\mathbb{N}}$ $x_j\in\mathbb{R}\quad\forall j\in\mathbb{N}$, each of which payable to t_j. Let $v(t,s)$ be the value function and $W(t,\boldsymbol{x})=\sum_{j=1}^{+\infty}x_j v(t,t_j)$.

Also a sequence of monetary items $\boldsymbol{y}=(y_j)_{j\in\mathbb{N}}$ be assigned with $y_j\in\mathbb{R}\quad\forall j\in\mathbb{N}$, each of which represents a single-payment due at time t_j. Let $w(t,s)$ be the value function and $W(t,\boldsymbol{y})=\sum_{j=1}^{+\infty}y_j w(t,t_j)$.

The following hypotheses be verified:

1. $\exists W(t,\boldsymbol{x})$, $\exists W(t,\boldsymbol{y})$ *and let us have* $W(t,\boldsymbol{x}) = W(t,\boldsymbol{y})$*;*
2. $\exists D(t,\boldsymbol{x})$, $\exists D(t,\boldsymbol{y})$ *and let us have* $D(t,\boldsymbol{x}) = D(t,\boldsymbol{y})$*;*
3. $\exists D^{(2)}(t,\boldsymbol{x})$, $\exists D^{(2)}(t,\boldsymbol{y})$ *and let us have* $D^{(2)}(t,\boldsymbol{x}) < D^{(2)}(t,\boldsymbol{y})$*;*
4. let z be an admissible additive shift;

Thesis:

$$\exists a > 0: \quad \forall z \in [-a,a], \ \text{admissible, it results:} \quad W(t,\boldsymbol{x},z) \leq W(t,\boldsymbol{y},z).$$

Proof. To get the verification it is necessary to consider the net actual value $W(t,\mathbf{x}-\mathbf{y},z)$ together with a reasoning process similar the one used to verify Redington's theorem. ♢

If we had:

$$D^{(2)}(t,\mathbf{x}) = D^{(2)}(t,\mathbf{y})$$

the two results mentioned above, are not conditions sufficient to decide whether there is immunization. In this case, it is necessary to consider also average durations of an order higher than the second.

Theorem 18 (REDINGTON (generalization)). *Let an issue date t, a sequence of maturities* $\boldsymbol{t} = (t_j)_{j\in\mathbb{N}}$*, with* $t < t_1$*,* $t_j < t_{j+1}$ $\forall j \in \mathbb{N}$*, a sequence of monetary items* $\boldsymbol{x} = (x_j)_{j\in\mathbb{N}}$ *be assigned with* $x_j \in \mathbb{R}$ $\forall j \in \mathbb{N}$*, each of which payable at time* t_j*. Let* $v(t,s)$ *be the value function and* $W(t,\boldsymbol{x}) = \sum_{j=1}^{+\infty} x_j v(t,t_j)$*.*

Let also a sequence of monetary items $\boldsymbol{y} = (y_j)_{j\in\mathbb{N}}$ *be assigned with* $y_j \in \mathbb{R}$ $\forall j \in \mathbb{N}$*, each of which being a single-payment due at time* t_j*. Let* $w(t,s)$ *be the value discount function and* $W(t,\boldsymbol{y}) = \sum_{j=1}^{+\infty} y_j w(t,t_j)$*.*

Let $h \in \mathbb{N}$ *with* $h \geq 2$ *be assigned such that* $\forall k = 1,2,...h \ \exists\, D^{k)}(t,\boldsymbol{x})$ *and* $\exists\, D^{(k)}(t,\boldsymbol{y})$
The following hypotheses be verified:

1. $W(t,\boldsymbol{x}) = W(t,\boldsymbol{y})$*;*
2. $D^{(k)}(t,\boldsymbol{x}) = D^{(k)}(t,\boldsymbol{y}) \quad \forall k = 1,...,h-1,$
3. $D^{(h)}(t,\boldsymbol{x}) \neq D^{(h)}(t,\boldsymbol{y})$.
4. let z be an admissible additive shift;

Thesis:

i) let h be a par number and $D^{(h)}(t,\boldsymbol{x}) > D^{(h)}(t,\boldsymbol{y})$ *then* $\exists a > 0$ *such that:*

$$\forall z \in [-a,a] \quad W(t^+,\boldsymbol{x},z) \geq W(t^+,\boldsymbol{y},z),$$

ii) let h be a par number and $D^{(h)}(t,\boldsymbol{x}) < D^{(h)}(t,\boldsymbol{y})$ *then* $\exists a > 0$ *such that:*

$$\forall z \in [-a,a] \quad W(t^+,\boldsymbol{x},z) \leq W(t^+,\boldsymbol{y},z),$$

iii) let h be an odd number and $D^{(h)}(t,\boldsymbol{x}) > D^{(h)}(t,\boldsymbol{y})$ *then* $\exists a > 0$ *such that:*

$$\begin{cases} \forall z \in [-a,0[\quad W(t^+,\boldsymbol{x},z) \geq W(t^+,\boldsymbol{y},z) \\ \\ \forall z \in]0,a] \quad W(t^+,\boldsymbol{x},z) \leq W(t^+,\boldsymbol{y},z) \end{cases}$$

iv) let h be an odd number and $D^{(h)}(t,\boldsymbol{x}) < D^{(h)}(t,\boldsymbol{y})$ *then* $\exists a > 0$ *such that:*

$$\begin{cases} \forall z \in [-a,0[\quad W(t^+,\boldsymbol{x},z) \leq W(t^+,\boldsymbol{y},z) \\ \\ \forall z \in]0,a] \quad W(t^+,\boldsymbol{x},z) \geq W(t^+,\boldsymbol{y},z) \end{cases}$$

Proof. The net actual value function is to be considered.

$$W(t,\mathbf{x}-\mathbf{y},z) = \sum_{j=1}^{+\infty} \left[x_j v(t,t_j) - y_j w(t,t_j)\right] e^{-z(t_j-t)}.$$

For the above mentioned hypotheses we have $W(t,\mathbf{x}-\mathbf{y},z) = 0$. Moreover, it is derivable, from order 1 until order h, for all z admissible and $\forall k = 1,\ldots,h-1$:

$$\frac{\partial^k W}{\partial z^k}(t,\mathbf{x}-\mathbf{y},0) = (-1)^k \left[W(t,\mathbf{x},0)D^{(k)}(t,\mathbf{x}) - W(t,\mathbf{y},0)D^{(k)}(t,\mathbf{y})\right] = 0$$

and we have:

$$\frac{\partial^h W}{\partial z^h}(t,\mathbf{x}-\mathbf{y},0) = (-1)^h \left[W(t,\mathbf{x},0)D^{(h)}(t,\mathbf{x}) - W(t,\mathbf{y},0)D^{(h)}(t,\mathbf{y})\right] \neq 0$$

from which, with some considerations typically introduced in the analysis about the existence of minimum point and maximum point, we deduce the thesis. ♢

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