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Asymmetric Information in a Market with $n+1$ Brownian Motions

Asimmetria Informativa in un Mercato con $n+1$ Moti Browniani

Luigi Romano and Donato Scolozzi

Abstract This paper covers asymmetric information in financial markets from a micro perspective. Particularly, we aim to extend the asset pricing framework introduced by Guasoni [2], who analyzes models price dynamics with both a martingale component, described by permanent shocks, and a stationary component, given by temporary shocks. First, we derive a generalization of this asset pricing model using n Brownian Motions, including an Ornstein-Uhlenbeck process as the $(n+1)th$ element. We find non-Markovian dynamics for the partially informed agents, which questions the validity of the efficient market hypothesis. Moreover, we compare the positions of informed and partially informed agents. Thereby, the filtration for informed agents is larger and initially specified, whereas the filtration for partially informed agents is smaller and obtained from the Hitsuda representation [3]. For both agents, our study yields similar results as the findings of Guasoni, for the logarithmic utility maximization problem.

Abstract *Questo lavoro esamina l'asimmetria informativa nei mercati finanziari applicabile anche ad una micro prospettiva. In particolare, ci proponiamo di estendere il lavoro sull'asset pricing introdotto da Guasoni [2], il quale analizza le dinamiche dei prezzi che presentano sia una componente martingala, descritta da shocks permanenti, sia una componente stazionaria, descritta da shocks temporanei. Inizialmente, deriviamo una generalizzazione di questo modello sull'asset pricing, utilizzando n Moti Browniani, prevedendo come $(n+1)th$ elemento un processo Ornstein-Uhlenbeck. Otteniamo una dinamica non Markoviana per gli agenti parzialmente informati, mettendo in tal modo in discussione la validitá delle ipotesi di mercato efficiente. Inoltre, confrontiamo le posizioni degli agenti informati con quelle degli agenti parzialmente informati. In questo quadro, la filtrazione per gli agenti parzialmente informati é piú grande e inizialmente assegnata, mentre la filtrazione per gli agenti non informati é piú piccola e ottenuta attraverso la rappresentazione di Hitsuda [3]. Per entrambi gli agenti,nell'ambito del problema della massimizzazione dell'utilitá logaritmica, i nostri studi forniscono risultati simili a quelli ottenuti da Guasoni.*

Key words: Stochastic Process, Hitsuda representation, Asymmetric information.

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1 The model

We consider a financial market where we have a riskless asset D togheter with a risky asset S. The market interest rate is assumed to be deterministic. In order to describe the dynamics of the risky asset, we consider a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ on which $n+1$ are defined, with $n \in$, independent Brownian Motions: $\left(B_t^1\right)_{t\in[0,+\infty[}$, $\left(B_t^2\right)_{t\in[0,+\infty[}$, . . . $\left(B_t^n\right)_{t\in[0,+\infty[}$, $\left(B_t^{n+1}\right)_{t\in[0,+\infty[}$.

If we set the real parameter $\lambda_{n+1} > 0$, we consider the Ornstein-Uhlenbeck process $\left(U_t^{n+1}\right)_{t\in[0,+\infty)}$ defined by the following equation:

$$U_t^{n+1} + \lambda_{n+1}\int_0^t U_s^{n+1} ds = B_t^{n+1}, \quad t \in [0,+\infty[\tag{1}$$

which, as known, is given by the following relation:

$$U_t^{n+1} = \int_0^t e^{-\lambda_{n+1}(t-s)} dB_s^{n+1}. \tag{2}$$

Then, if we set the real numbers p_j, with $j = 1,2,...,n,n+1$, $p_{n+1} > 0$, with the first n numbers not all zero, let us consider the process $(Y_t)_{t\in[0,+\infty)}$ defined by:

$$Y_t = \sum_{j=1}^{n} p_j B_t^j + p_{n+1} U_t^{n+1}. \tag{3}$$

Now, let us introduce two deterministic Lebesgue measurable functions

$$\mu, \sigma : [0,+\infty[\longrightarrow [0,+\infty[$$

such that

$$\forall\, T > 0 \qquad \mu \in L^1([0,T]), \quad \sigma \in L^2([0,T]).$$

Suppose that the price of the risky asset is described by the following differential equation:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dY_t \tag{4}$$

whose solution, as known, is given by the relation

$$S_t = S_0 \exp\left[\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dY_s\right]. \tag{5}$$

Now we can describe the previous situation in the following way: we have an ”informed agent” who has all the information provided by the all Brownian Motions, and a ”partially informed agent” who has all the information provided by the process Y_t. The informed agent refers to the filtration $\left(\mathscr{F}_t^1\right)_{t\in[0,+\infty[}$ obtained by completing the natural filtration generated by $n+1$ Brownian Motions $B_t^1, B_t^2, \ldots, B_t^n, B_t^{n+1}$, which therefore satisfies the usual conditions of completeness and right continuity. The partially informed agent, instead, refers to the filtration $\left(\mathscr{F}_t^0\right)_{t\in[0,+\infty[}$ generated by the process Y_t. Of course, we have that $\mathscr{F}_t^0 \subset \mathscr{F}_t^1, \ \forall\, t$. We might state that the informed agent's risky asset value evolves according to the assigned Brownian Motions; therefore its value is determined by the following equation:

$$\frac{dS_t}{S_t} = \left(\mu_t - p_{n+1}\lambda_{n+1}\sigma_t U_t^{n+1}\right) dt + \sigma_t \sum_{j=1}^{n+1} p_j dB_t^j \tag{6}$$

which refers to the Brownian Motion

$$W_t = \left[\sum_{j=1}^{n+1} p_j^2\right]^{-\frac{1}{2}} \sum_{j=1}^{n+1} p_j B_t^j$$

whose solution, if $x > 0$ is the initial wealth, is given, as known, by the relation:

$$S_t = xe^{\int_0^t \left[\left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\sigma_s^2\right] ds + \left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \int_0^t \sigma_s dW_s}. \quad (7)$$

In the next section, we want to derive, for the partially informed agent, an analogous equation which represents Y_t, and therefore S, in terms of filtration $\mathscr{F}^0$ and of an opportune Brownian Motion B^0.

2 Factorization of the Process Y_t with respect to $\mathscr{F}^0$.

In this section, we shall examine the Markov property of Y_t and will determine, with respect to the filtration $\mathscr{F}^0$, the relative Brownian Motion which represents it.

Theorem 1. *Y_t is a Gaussian process, and moreover:*

1. $E(Y_t) = 0 \quad \forall t \in [0, +\infty[$
2. $\Gamma(s,t) = cov(Y_s, Y_t) = \left(\sum_{j=1}^{n} p_j^2\right) t \wedge s + p_{n+1}^2 \frac{e^{-\lambda_{n+1}|t-s|} - e^{-\lambda_{n+1}(t+s)}}{2\lambda_{n+1}}$.

Proof. 1. Gaussian and mean zero properties are obvious. Besides:
2. $\Gamma(s,t) = cov(Y_s, Y_t) = E(Y_s Y_t) =$
$E\left(\left[\sum_{j=1}^{n} p_j B_s^j + p_{n+1} U_s^{n+1}\right]\left[\sum_{j=1}^{n} p_j B_t^j + p_{n+1} U_t^{n+1}\right]\right) =$
utilizing the independence property of the Brownian Motion, we have that:
$= E\left(\sum_{j=1}^{n} p_j^2 B_s^j B_t^j\right) + E\left(p_{n+1}^2 U_s^{n+1} U_t^{n+1}\right) =$
$\sum_{j=1}^{n} p_j^2 E\left(B_s^j B_t^j\right) + p_{n+1}^2 E\left(\int_0^s e^{-\lambda_{n+1}(s-u)} dB_u^{n+1} \int_0^t e^{-\lambda_{n+1}(t-u)} dB_u^{n+1}\right) =$
$\left(\sum_{j=1}^{n} p_j^2\right) s \wedge t + p_{n+1}^2 \int_0^{s\wedge t} e^{-\lambda_{n+1}(t-u) - \lambda_{n+1}(s-u)} du =$
$\left(\sum_{j=1}^{n} p_j^2\right) s \wedge t + p_{n+1}^2 e^{-\lambda_{n+1}(t+s)} \frac{e^{2\lambda_{n+1}(s\wedge t)} - 1}{2\lambda_{n+1}} =$
$\left(\sum_{j=1}^{n} p_j^2\right) s \wedge t + p_{n+1}^2 \frac{e^{-\lambda_{n+1}|t-s|} - e^{-\lambda_{n+1}(t+s)}}{2\lambda_{n+1}}$

To verify the Markov property of the process Y_t, we recall the following result [4] (III.1.13)

Theorem 2. *Y_t is a Markov process if, and only if, we have:*

$$\Gamma(s,t)\Gamma(t,u) = \Gamma(t,t)\Gamma(s,u), \quad \forall\, s \le t \le u.$$

Theorem 3. *1. If we assume that $\lambda_{n+1} = 0$, then Y_t is a Markov process.*

2. *If we assume that $\lambda_{n+1} > 0$ then Y_t is a Markov process if, and only if, we have one of the two following conditions:* $p_j = 0 \quad \forall j = 1, 2, ..., n$ *or* $p_{n+1} = 0$.

Proof. Property 1 is obvious. Besides it is obvious that Y_t is a Markov process if we have $p_j = 0 \quad \forall j = 1, 2, ..., n$ or if we have $p_{n+1} = 0$.

Then let us have that:

$$\Gamma(s,t)\Gamma(t,u) = \Gamma(t,t)\Gamma(s,u) \quad \forall\, s \leq t \leq u$$

and also suppose that $\sum_{j=1}^{n} p_j^2 > 0$.

Considering the limit for $u \to +\infty$ of

$$\Gamma(s,t)\Gamma(t,u)$$

and

$$\Gamma(t,t)\Gamma(s,u)$$

we have:

$$t p_{n+1}^2 \frac{e^{-\lambda_{n+1}(t-s)} - e^{-\lambda_{n+1}(t+s)}}{2\lambda_{n+1}} = s p_{n+1}^2 \frac{1 - e^{-2\lambda_{n+1} t}}{2\lambda_{n+1}}$$

which can be written also as follows

$$p_{n+1}^2 \left[\frac{e^{\lambda_{n+1} s} - e^{-\lambda_{n+1} s}}{s} - \frac{e^{\lambda_{n+1} t} - e^{-\lambda_{n+1} t}}{t} \right] = 0, \quad \forall s \leq t$$

from which, considering the limit for $t \to +\infty$, we deduce the thesis: $p_{n+1} = 0$.

Now let us consider the process Z defined by the relation:

$$Z_t = \left(\sum_{j=1}^{n+1} p_j^2 \right)^{-\frac{1}{2}} \sum_{j=1}^{n} p_j \left(B_t^j + \lambda_{n+1} \int_0^t B_u^j du \right) + \left(\sum_{j=1}^{n+1} p_j^2 \right)^{-\frac{1}{2}} p_{n+1} B_t^{n+1}. \tag{8}$$

It verifies the following result:

Theorem 4. *1. Z_t is a Gaussian process.*
2. $E(Z_t) = 0 \quad \forall t \in [0, +\infty[$.
3. $cov(Z_t, Z_s) = t \wedge s + \left(\sum_{j=1}^{n+1} p_j^2 \right)^{-1} \left(\sum_{j=1}^{n} p_j^2 \right) \int_0^t \int_0^s \left(\lambda_{n+1} + \lambda_{n+1}^2 u \wedge v \right) du dv$.

Proof. We note that the Z process can be re-written in the form:

$$Z_t = \left(\sum_{j=1}^{n+1} p_j^2 \right)^{-\frac{1}{2}} \sum_{j=1}^{n} p_j \left(B_t^j + \lambda_{n+1} \int_0^t (t-u)\, dB_u^j \right) + \left(\sum_{j=1}^{n+1} p_j^2 \right)^{-\frac{1}{2}} p_{n+1} B_t^{n+1} =$$

$$\left(\sum_{j=1}^{n+1} p_j^2 \right)^{-\frac{1}{2}} \sum_{j=1}^{n} p_j \int_0^t \left[1 + \lambda_{n+1} (t-u) \right] dB_u^j + \left(\sum_{j=1}^{n+1} p_j^2 \right)^{-\frac{1}{2}} p_{n+1} B_t^{n+1}.$$

Therefore the covariance, because of the independence of the Brownian Motions, is given by:

$$cov\left(Z_t, Z_s\right) = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-1} \sum_{j=1}^{n} p_j^2 \int_0^{t\wedge s} \left[1+\lambda_{n+1}\left(t-u\right)\right]\left[1+\lambda_{n+1}\left(s-u\right)\right] du +$$
$$+ \left(\sum_{j=1}^{n+1} p_j^2\right)^{-1} p_{n+1}^2 \, t\wedge s$$

by means of standard calculations, we get the final relation.

Now let us consider the following function

$$\tilde{f}\left(t,s\right) = -\left(\sum_{j=1}^{n+1} p_j^2\right)^{-1}\left(\sum_{j=1}^{n} p_j^2\right)\left(\lambda_{n+1} + \lambda_{n+1}^2 t\wedge s\right)$$

which is part of the covariance of the process Z_t. For our further aims, if $0 \le s \le t$, then the formula can also be written as follows:

$$\tilde{f}\left(t,s\right) = -\left(\sum_{j=1}^{n+1} p_j^2\right)^{-1}\left(\sum_{j=1}^{n} p_j^2\right)\left(\lambda_{n+1} + \lambda_{n+1}^2 s\right) \quad \forall\, 0 \le s \le t.$$

To simplify, if $A^2 = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-1}\left(\sum_{j=1}^{n} p_j^2\right)$, we can consider the following result:

Theorem 5. *Considering the previous function $\tilde{f}\left(t,s\right)$, the function*

$$\tilde{g}\left(t,s\right) = \begin{cases} \lambda_{n+1}\eta\left(s\right) & \text{for } 0 \le s \le t \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

verifies the following integral equation

$$\tilde{f}\left(t,s\right) = \tilde{g}\left(t,s\right) - \int_0^s \tilde{g}\left(t,u\right)\tilde{g}\left(s,u\right) du \quad \forall\, 0 \le s \le t \tag{10}$$

and $\eta\left(s\right)$ verifies the following Cauchy problem

$$\begin{cases} \eta'\left(s\right) = \lambda_{n+1}\left(\eta\left(s\right) - A^2\right) \\ \eta\left(0\right) = -A^2. \end{cases} \tag{11}$$

Proof. It is easy to verify this, considering the following integral equation:

$$-A^2\left(\lambda_{n+1} + \lambda_{n+1}^2 s\right) = \lambda_{n+1}\eta\left(s\right) - \lambda_{n+1}^2 \int_0^s \eta^2\left(u\right) du \quad \forall\, 0 \le s \le t$$

from which we easily obtain the Cauchy problem.
Its solution, as already verified, is given by the function:

$$\eta\left(s\right) = A\frac{1-A-\left(1+A\right)e^{2A\lambda_{n+1}s}}{1-A+\left(1+A\right)e^{2A\lambda_{n+1}s}}.$$

At this point we are able to enunciate the following theorem:

Theorem 6. *1. Consider a Brownian Motion $\left(B_t^0\right)_{t\in[0,+\infty[}$ with respect to the filtration $\left(\mathscr{F}_t^0\right)_{t\in[0,+\infty[}$ such that we have:*

$$Z_t = B_t^0 - \int_0^t \left(\int_0^s \tilde{g}(s,u)\, dB_u^0 \right) ds = B_t^0 - \lambda_{n+1} \int_0^t \left(\int_0^s \eta(u)\, dB_u^0 \right) ds.$$

2. *Considering the function* $g : [0,T] \times [0,T] \longrightarrow \Re$ *defined by*

$$g(t,s) = \begin{cases} -\lambda_{n+1}\eta(s) e^{\lambda_{n+1}\int_s^t \eta(u)du} & \text{for } 0 \le s \le t \\ 0 \quad \text{otherwise} \end{cases} \tag{12}$$

we obtain that:

$$B_t^0 = Z_t - \int_0^t \left(\int_0^s g(s,u)\, dZ_u \right) ds.$$

Proof. 1. The existence of the Brownian Motion B_t^0 is a consequence of [3], proposition 2, and also of the fact that the function $\tilde{g}(t,s)$ verifies the following integral equation:

$$\tilde{f}(t,s) = \tilde{g}(t,s) - \int_0^s \tilde{g}(t,u)\, \tilde{g}(s,u)\, du \quad \forall\, 0 \le s \le t.$$

2. To verify the relation

$$B_t^0 = Z_t - \int_0^t \left(\int_0^s g(s,u)\, dZ_u \right) ds$$

it is sufficient to utilize [1] or [2] or [3]. The function $g(t,s)$ is called the negative resolvent of $\tilde{g}(t,s)$.

Theorem 7. *The processes* Y_t *and* Z *verify the following equation:*

$$Y_t + \lambda_{n+1} \int_0^t Y_u du = \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} Z_t$$

so we have

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} \int_0^t e^{-\lambda_{n+1}(t-u)} dZ_u.$$

Proof. We have that:

$$Y_t - \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} Z_t = \sum_{j=1}^{n+1} p_j B_t^j + p_{n+1} U_t^{n+1} - \sum_{j=1}^{n+1} p_j \left(B_t^j + \lambda_{n+1} \int_0^t B_u^j du \right) - p_{n+1} B_t^{n+1} =$$

$$-\lambda_{n+1} \int_0^t \left[\sum_{j=1}^{n+1} p_j B_u^j + p_{n+1} U_u^{n+1} \right] du = -\lambda_{n+1} \int_0^t Y_u du.$$

By integration we easily obtain the second relation.

It is now possible to establish the link between the process Y_t and the Brownian Motion B_t^0. Namely, we have the following (fundamental) result:

Theorem 8. *Let $Y_t = \sum_{j=1}^{n} p_j B_t^j + p_{n+1} U_t^{n+1}$ and $\left(\mathscr{F}_t^0\right)_{t\in[0,+\infty[}$ be its completed natural filtration. As we have already noted, B_t^0 is a Brownian Motion with respect to the filtration $\left(\mathscr{F}_t^0\right)_{t\in[0,+\infty[}$ of Y_t. Besides, we suppose that:*

1.
$$B_t^0 = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} \left[Y_t + \lambda_{n+1}\int_0^t \left(\int_0^s \left[1+\eta(u)\right] e^{\lambda_{n+1}\int_u^s \eta(l)dl} dY_u\right) ds\right]$$

2.
$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \int_0^t \left[e^{-\lambda_{n+1}(t-u)}\left[1+\eta(u)\right] - \eta(u)\right] dB_u^0$$

3.
$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \left[B_t^0 - \lambda_{n+1}\int_0^t \left[e^{-\lambda_{n+1}(t-u)}\left(B_u^0 + \int_0^u \eta(v) dB_v^0\right)\right] du\right]$$

4.
$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \left[B_t^0 - \lambda_{n+1}\int_0^t \left(\int_0^s \left[1+\eta(u)\right] e^{-\lambda_{n+1}(s-u)} dB_u^0\right) ds\right].$$

Proof. 1. In order to obtain the first relation, consider

$$B_t^0 = Z_t - \int_0^t \left(\int_0^s g(s,u)\, dZ_u\right) ds$$

in which we substitute Z_t for the following equation

$$Z_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} \left[Y_t + \lambda_{n+1}\int_0^t Y_u du\right]$$

so that we obtain:

$$B_t^0 = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} \left[Y_t + \int_0^t \left(\lambda_{n+1} Y_s - \int_0^s g(s,u) dY_u - \int_0^s g(s,u) Y_u du\right) ds\right].$$

Utilizing the method of integration by parts in the following integral

$$\int_0^s g(s,u) Y_u du$$

and if we suppose $G(s,u) = \int_0^u g(s,v)dv$, it is easy to obtain the relation:

$$\int_0^s g(s,u) Y_u du = \lambda_{n+1} Y_s - \int_0^s G(s,u) dY_u$$

substituting in B_t^0, we obtain the relation 1.

2. In the relation

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \int_0^t e^{-\lambda_{n+1}(t-s)} dZ_s$$

we substitute Z_u with the relation

$$Z_t = B_t^0 - \lambda_{n+1} \int_0^t \left(\int_0^s \eta\,(u)\, dB_u^0 \right) ds$$

so that we obtain

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \left[\int_0^t e^{-\lambda_{n+1}(t-s)} dB_s^0 - \lambda_{n+1} \int_0^t \left(\int_0^s \eta(u) e^{-\lambda_{n+1}(t-s)} dB_u^0 \right) ds \right]$$

which, applying the Fubini Tonelli theorem, can be written in the form:

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \left[\int_0^t e^{-\lambda_{n+1}(t-s)} dB_s^0 - \lambda_{n+1} \int_0^t \left(\int_u^t \eta(u) e^{-\lambda_{n+1}(t-s)} ds \right) dB_u^0 \right]$$

which, simplified, gives the relation 2.

3. In order to obtain the above relation, we consider the equation

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \left[\int_0^t e^{-\lambda_{n+1}(t-s)} dB_s^0 - \lambda_{n+1} \int_0^t \left(\int_0^s \eta(u) e^{-\lambda_{n+1}(t-s)} dB_u^0 \right) ds \right]$$

utilizing the method of integration by parts on the first integral, we obtain:

$$\int_0^t e^{-\lambda_{n+1}(t-s)} dB_s^0 = B_t^0 - \lambda_{n+1} \int_0^t e^{-\lambda_{n+1}(t-s)} B_s^0 ds.$$

Substituting and simplifying we obtain the result 3.

4. In equation 1, if $W_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} Y_t$, we have

$$B_t^0 = W_t + \lambda_{n+1} \int_0^t \left(\int_0^s [1+\eta(u)]\, e^{\lambda_{n+1} \int_u^s \eta(l) dl} dW_u \right) ds$$

which, written in standard form

$$B_t^0 = W_t - \int_0^t \left(\int_0^s -\lambda_{n+1} [1+\eta(u)]\, e^{\lambda_{n+1} \int_u^s \eta(l) dl} dW_u \right) ds$$

identifies the following Volterra Kernel

$$k(t,s) = \begin{cases} -\lambda_{n+1} [1+\eta(s)]\, e^{\lambda_{n+1} \int_s^t \eta(l) dl} & \text{for } 0 \le s \le t \\ 0 \quad \text{otherwise.} \end{cases}$$

Utilizing [2], we also identify the relative negative resolvent $\tilde{k}(t,s)$ through the relation

$$\tilde{k}(t,s) = \begin{cases} -k(t,s)e^{\int_s^t k(u,u)du} & \text{for } 0 \le s \le t \\ 0 & \text{otherwise} \end{cases}$$

from which, by substitution, we obtain the following relation:

$$\tilde{k}(t,s) = \begin{cases} \lambda_{n+1}\left[1+\eta(s)\right]e^{-\lambda_{n+1}(t-s)} & \text{se } 0 \le s \le t \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence we have

$$W_t = B_t^0 - \int_0^t \left(\int_0^s \tilde{k}(s,u)dB_u^0 \right) ds$$

from which, we deduce relation 4 :

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} \left[B_t^0 - \lambda_{n+1} \int_0^t \left(\int_0^s \left[1+\eta(u)\right] e^{-\lambda_{n+1}(s-u)} dB_u^0 \right) ds \right].$$

On the basic of result 4 of the previous theorem, the asset price dynamics for the partially informed agent, can be shown evolving as follows:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} \left[dB_t^0 - \lambda_{n+1} \left(\int_0^t \left[1+\eta(u)\right] e^{-\lambda_{n+1}(t-u)} dB_u^0 \right) dt \right] \tag{13}$$

which can be re-written in the form:

$$\frac{dS_t}{S_t} = \left[\mu_t - \lambda_{n+1} \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} \left(\sigma_t \int_0^t \left[1+\eta(u)\right] e^{-\lambda_{n+1}(t-u)} dB_u^0 \right) \right] dt + \sigma_t \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} dB_t^0. \tag{14}$$

Conversely, for the informed agent, the asset price dynamics can be shown evolving in the following equation:

$$\frac{dS_t}{S_t} = \left(\mu_t - p_{n+1}\lambda_{n+1}\sigma_t U_t^{n+1} \right) dt + \sigma_t \sum_{j=1}^{n+1} p_j dB_t^j \tag{15}$$

which refers to the Brownian Motion

$$\left[\sum_{j=1}^{n+1} p_j^2 \right]^{-\frac{1}{2}} \sum_{j=1}^{n+1} p_j B_t^j.$$

3 The value functions for two agents.

As already said in the previous section, the informed agent considers the underlying value starting from an initial wealth $x > 0$, and investing H_t units of S_t. He obtains the self-financed value of wealth X_t, at time t, through all the assigned Brownian Motion. Conversely the partially informed agent, invests the same monetary item $x > 0$, and he utilizes the Brownian Motion B_t^0 in order to assess the dynamics of the wealth obtained. In this section, we want to examine these two situations, and we also want to evaluate the utility functions the two agents use.

3.1 The value function for the informed agent.

Let $x > 0$ be the initial monetary item that the partially informed agent invests in asset S_t. To do this, he utilizes an opportune stochastic process H_t which, at time t, represents the asset shares used. So he obtains the value (self-financed) of wealth X_t, at time t, through the following relation:

$$X_t = x + \int_0^t H_s dS_s. \tag{16}$$

As already said, the process H_t, which will be said admissible, must be predictable with respect to filtration $\left(\mathscr{F}_t^1\right)_{t\in[0,+\infty[}$, integrable with respect to process S_t and such that almost certainly we also have $X_t > 0$, $\forall t \in [0,T]$.

Finally, if $\mathscr{U}$ is the utility function, the agent maximizes the mean utility of the wealth obtained in the final instant T. Thereby it solves the following problem:

$$\sup\left\{ E\left(\mathscr{U}\left(x+\int_0^T H_t dS_t\right)\right) : \quad H_t \quad admissible \right\}. \tag{17}$$

In order to guarantee the positivity of the wealth produced at every instant t, we can consider the process π_t defined by the relation $H_t = \pi_t \frac{X_t}{S_t}$. Therefore we have:

$$X_t = x + \int_0^t \pi_s \frac{X_s}{S_s} dS_s \tag{18}$$

from which the deduction

$$\frac{dX_t}{X_t} = \pi_t \frac{dS_t}{S_t} \tag{19}$$

that is to say

$$\frac{dX_t}{X_t} = \pi_t \left(\mu_t - p_{n+1}\lambda_{n+1}\sigma_t U_t^{n+1}\right) dt + \pi_t \sigma_t \sum_{j=1}^{n+1} p_j dB_t^j. \tag{20}$$

If we isolate the Brownian Motion:

$$W_t = \left[\sum_{j=1}^{n+1} p_j^2\right]^{-\frac{1}{2}} \sum_{j=1}^{n+1} p_j B_t^j$$

we have:

$$\frac{dX_t}{X_t} = \pi_t \left(\mu_t - p_{n+1}\lambda_{n+1}\sigma_t U_t^{n+1}\right) dt + \left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \pi_t \sigma_t dW_t. \tag{21}$$

Therefore the value of wealth X_t, at time t, is given by the relation

$$X_t = xe^{\int_0^t \left[\pi_s \left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\pi_s^2\sigma_s^2\right] ds + \left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \int_0^t \pi_s \sigma_s dW_s} \tag{22}$$

and, as a consequence, at final instant T we have:

$$X_T = xe^{\int_0^T \left[\pi_s \left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\pi_s^2\sigma_s^2\right] ds + \left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \int_0^T \pi_s\sigma_s dW_s}. \tag{23}$$

Now, considering the logarithmic utility function, we have the following result:

Theorem 9. *Let $\mathscr{U}(y) = \log y$. The process*

$$\pi_s = \frac{\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}}{\left(\sum_{j=1}^{n+1} p_j^2\right)\sigma_s^2}$$

provides the optimal investment share and the relative value function is given by:

$$u(x) = \log x + \frac{1}{2\left(\sum_{j=1}^{n+1} p_j^2\right)} E\left[\int_0^T \frac{\left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right)^2}{\sigma_s^2} ds\right].$$

Proof. $\forall\ \pi_s$ admissible, it results

$$\mathscr{U}(X_T) = \log(X_T) = \log x + \int_0^T \left[\pi_s \left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\pi_s^2\sigma_s^2\right] ds +$$

$$\left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \int_0^T \pi_s\sigma_s dW_s$$

as a consequence, considering the mean value, we have:

$$E\left(\log(X_T)\right) = \log x + E\left[\int_0^T \left[\pi_s \left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\pi_s^2\sigma_s^2\right] ds\right].$$

We obtain the relative maximun value, applying the results that allow to derive an integral. As a consequence the portfolio share is given by

$$\pi_s = \frac{\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}}{\left(\sum_{j=1}^{n+1} p_j^2\right)\sigma_s^2}$$

therefore the relative maximum value is:

$$u(x) = \log x + \frac{1}{2} E\left[\int_0^T \frac{\left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right)^2}{\left(\sum_{j=1}^{n+1} p_j^2\right)\sigma_s^2} ds\right]$$

from which we deduce the relation looked for.

We can re-write the value function, in the following way:

Theorem 10. *We have:*

$$u(x)=\log x+\frac{1}{2\sum_{j=1}^{n+1}p_j^2}\int_0^T\frac{\mu_s^2}{\sigma_s^2}ds+\frac{p_{n+1}^2}{\sum_{j=1}^{n+1}p_j^2}\left[\frac{T\lambda_{n+1}}{4}-\frac{1-e^{-2T\lambda_{n+1}}}{8}\right].$$

Proof. From the previous equation we have:

$$u(x)=\log x+\frac{1}{2\sum_{j=1}^{n+1}p_j^2}E\left[\int_0^T\frac{\mu_s^2}{\sigma_s^2}ds\right]-\frac{p_{n+1}\lambda_{n+1}}{\sum_{j=1}^{n+1}p_j^2}E\left[\int_0^T\frac{\mu_s}{\sigma_s}U_s^{n+1}ds\right]+\frac{p_{n+1}^2\lambda_{n+1}^2}{2\sum_{j=1}^{n+1}p_j^2}E\left[\int_0^T\left[U_s^{n+1}\right]^2ds\right].$$

Since the functions μ_s and σ_s are deterministic, we have:
$E\left[\int_0^T\frac{\mu_s^2}{\sigma_s^2}ds\right]=\int_0^T\frac{\mu_s^2}{\sigma_s^2}ds.$

Moreover it results:
$E\left[\int_0^T\frac{\mu_s}{\sigma_s}U_s^{n+1}ds\right]=E\left[\int_0^T\left(\frac{\mu_s}{\sigma_s}\int_0^s e^{-\lambda_{n+1}(s-u)}dB_u^{n+1}\right)ds\right]=\int_0^T E\left(\frac{\mu_s}{\sigma_s}\int_0^s e^{-\lambda_{n+1}(s-u)}dB_u^{n+1}\right)ds=$
$\int_0^T\frac{\mu_s}{\sigma_s}E\left(\int_0^s e^{-\lambda_{n+1}(s-u)}dB_u^{n+1}\right)ds=\int_0^T\frac{\mu_s}{\sigma_s}0ds=0.$

Finally, we have:
$E\left[\int_0^T\left[U_s^{n+1}\right]^2ds\right]=\int_0^T E\left[\left[\int_0^s e^{-\lambda_{n+1}(s-u)}dB_u^{n+1}\right]^2\right]ds=\int_0^T\left[\int_0^s e^{-2\lambda_{n+1}(s-u)}du\right]ds=\frac{T}{2\lambda_{n+1}}-\frac{1-e^{-2\lambda_{n+1}T}}{4\lambda_{n+1}^2}.$
In this way we have the thesis.

3.2 The value function for the partially informed agent.

Similarly, the partially informed agent considers the investment shares provided through processes K_t admissible: they are predictable with respect to the filtration $\left(\mathscr{F}_t^0\right)_{t\in[0,+\infty[}$, integrable with respect to the process S_t and such that $X_t>0$ almost certainly and $\forall t\in[0,T]$. If $\mathscr{V}$ is his utility function, then the agent maximizes his expected utility of wealth at time T. Therefore it solves the following problem:

$$\max\left\{E\left(\mathscr{V}\left(x+\int_0^T K_t dS_t\right)\right):\quad K_t\quad admissible\quad\right\}. \tag{24}$$

Also in this case the agent considers the process κ_t defined by the relation:

$$K_t=\kappa_t\frac{X_t}{S_t}.$$

Therefore we have:

$$X_t=x+\int_0^t\kappa_s\frac{X_s}{S_s}dS_s \tag{25}$$

from which the deduction

$$\frac{dX_t}{X_t}=\kappa_t\frac{dS_t}{S_t} \tag{26}$$

that is to say, if

$$\nu_t = -\lambda_{n+1}\int_0^t [1+\eta(u)]\, e^{-\lambda_{n+1}(t-u)} dB_u^0$$

we have:

$$\frac{dX_t}{X_t} = \kappa_t \left[\mu_t + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_t \sigma_t\right] dt + \kappa_t \sigma_t \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} dB_t^0. \tag{27}$$

The wealth at time t, if $x > 0$ is the initial one, is therefore given by

$$X_t = xe^{\int_0^t \left(\kappa_s \left[\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s \sigma_s\right] - \frac{1}{2}\kappa_s^2 \sigma_s^2 \left(\sum_{j=1}^{n+1} p_j^2\right)\right) ds + \int_0^t \kappa_s \sigma_s \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} dB_s^0} \tag{28}$$

and, as a consequence, at final time T we have:

$$X_T = xe^{\int_0^T \left(\kappa_s \left[\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s \sigma_s\right] - \frac{1}{2}\kappa_s^2 \sigma_s^2 \left(\sum_{j=1}^{n+1} p_j^2\right)\right) ds + \int_0^T \kappa_s \sigma_s \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} dB_s^0}. \tag{29}$$

Considering now the logarithmic utility function, we have the following result:

Theorem 11. *Let $\mathscr{V}(y) = \log y$. The process*

$$\kappa_s = \frac{\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s \sigma_s}{\left(\sum_{j=1}^{n+1} p_j^2\right) \sigma_s^2}$$

where

$$\nu_s = -\lambda_{n+1}\int_0^s [1+\eta(u)]\, e^{-\lambda_{n+1}(s-u)} dB_u^0$$

provides the optimal investment share. The relative value function is given by:

$$v(x) = \log x + \frac{1}{2\left(\sum_{j=1}^{n+1} p_j^2\right)} E\left[\int_0^T \frac{1}{\sigma_s^2}\left(\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s \sigma_s\right)^2 ds\right].$$

Proof. $\forall\ \kappa_s$ admissible, it results

$$\mathscr{V}(X_T) = \log(X_T) = \log x + \int_0^T \left[\kappa_s \left(\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s \sigma_s\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\kappa_s^2 \sigma_s^2\right] ds +$$

$$\left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \int_0^T \pi_s \sigma_s dW_s$$

as a consequence, considering the mean value, we obtain that:

$$E\left(\log(X_T)\right)=\log x+E\left[\int_0^T\left[\kappa_s\left(\mu_s+\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}\nu_s\sigma_s\right)-\frac{1}{2}\left(\sum_{j=1}^{n+1}p_j^2\right)\kappa_s^2\sigma_s^2\right]ds\right].$$

The relative maximum value is obtained applying the theorems which allow to derive an integral. As a consequence the portfolio share which maximizes the result is given by

$$\kappa_s=\frac{\mu_s+\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}\nu_s\sigma_s}{\left(\sum_{j=1}^{n+1}p_j^2\right)\sigma_s^2}$$

therefore the relative maximum value is:

$$v(x)=\log x+\frac{1}{2}E\left[\int_0^T\frac{\left(\mu_s+\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}\nu_s\sigma_s\right)^2}{\left(\sum_{j=1}^{n+1}p_j^2\right)\sigma_s^2}ds\right]$$

from which, we deduce the relation looked for.

We can re-write the value function, in the following way:

Theorem 12. *We have:*

$$v(x)=\log x+\frac{1}{2\sum_{j=1}^{n+1}p_j^2}\int_0^T\frac{\mu_s^2}{\sigma_s^2}ds+\frac{\lambda_{n+1}^2}{2}\int_0^T\left[\int_0^s\left[1+\eta\left(u\right)\right]^2e^{-2\lambda_{n+1}(s-u)}du\right]ds.$$

Proof. From the equation

$$v(x)=\log x+\frac{1}{2\left(\sum_{j=1}^{n+1}p_j^2\right)}E\left[\int_0^T\frac{1}{\sigma_s^2}\left(\mu_s+\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}\nu_s\sigma_s\right)^2ds\right]$$

we have:

$$v(x)=\log x+\frac{1}{2\sum_{j=1}^{n+1}p_j^2}E\left[\int_0^T\frac{\mu_s^2}{\sigma_s^2}ds\right]+\frac{1}{\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}}E\left[\int_0^T\frac{\mu_s}{\sigma_s}\nu_s ds\right]+\frac{1}{2}E\left[\int_0^T\nu_s^2ds\right]=$$

$$\log x+\frac{1}{2\sum_{j=1}^{n+1}p_j^2}\int_0^T\frac{\mu_s^2}{\sigma_s^2}ds+\frac{1}{\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}}\int_0^T\frac{\mu_s}{\sigma_s}E\left[\nu_s\right]ds+\frac{1}{2}E\left[\int_0^T\nu_s^2ds\right]=$$

$$\log x + \frac{1}{2\sum_{j=1}^{n+1} p_j^2} \int_0^T \frac{\mu_s^2}{\sigma_s^2} ds + \frac{1}{\left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}}} \int_0^T \frac{\mu_s}{\sigma_s} E\left[\nu_s\right] ds + \frac{1}{2} E\left[\int_0^T \nu_s^2 ds\right].$$

Besides it results that:
$E\left[\nu_s\right] = E\left[-\lambda_{n+1}\int_0^s \left[1+\eta(u)\right] e^{-\lambda_{n+1}(s-u)} dB_u^0\right] = 0.$

Finally we have:
$E\left[\int_0^T \nu_s^2 ds\right] = \int_0^T E\left[\nu_s^2\right] ds = \int_0^T E\left[\left(-\lambda_{n+1}\int_0^s \left[1+\eta(u)\right] e^{-\lambda_{n+1}(s-u)} dB_u^0\right)^2\right] ds =$
$\lambda_{n+1}^2 \int_0^T \left[\int_0^s \left[1+\eta(u)\right]^2 e^{-2\lambda_{n+1}(s-u)} du\right] ds.$

From the results obtained we have the thesis.

In order to compare the two value functions so far obtained, we consider the following theorem:

Theorem 13. *We have:*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \left[\int_0^s \left[1+\eta\left(u\right)\right]^2 e^{-2\lambda_{n+1}(s-u)} du\right] ds = \frac{(1-A)^2}{2\lambda_{n+1}}$$

where $A = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}}$.

Proof. First, we verify that

$$\lim_{T\to+\infty} \int_0^T \left[1+\eta\left(u\right)\right]^2 e^{-2\lambda_{n+1}(T-u)} du = \frac{(1-A)^2}{2\lambda_{n+1}}$$

if, within integral, we substitute $v = T-u$, we obtain:

$$\int_0^T \left[1+\eta\left(T-v\right)\right]^2 e^{-2\lambda_{n+1}(v)} dv.$$

Besides, if we denote the indicator function of the interval $[0,T]$ with $I_{[0,T]}$, we can write the limit as

$$\lim_{T\to+\infty} \int_0^{+\infty} \left[1+\eta\left(T-v\right)\right]^2 e^{-2\lambda_{n+1}(v)} I_{[0,T]}\left(v\right) dv.$$

Now, note that, since the $\lim\limits_{T\to+\infty} \eta\left(T-v\right) = -A$, the integrand function tends punctually to the function $v \in [0,+\infty[\mapsto (1-A)^2 e^{-2\lambda_{n+1}v}$. Moreover the integrand function verifies the following inequalities:

$$0 \le \left[1+\eta\left(T-v\right)\right]^2 e^{-2\lambda_{n+1}(v)} I_{[0,T]}\left(v\right) dv \le (1+A)\, e^{-2\lambda_{n+1}v} \quad \forall\, v \in [0,+\infty[$$

and the decrease and increase functions are integrable on $[0,+\infty[$. The Lebesgue dominated convergence theorem allows the following relation

$$\lim_{T\to+\infty} \int_0^{+\infty} \left[1+\eta\left(T-v\right)\right]^2 e^{-2\lambda_{n+1}(v)} I_{[0,T]}\left(v\right) dv = \int_0^{+\infty} (1-A)^2 e^{-2\lambda_{n+1}v} dv$$

from which, integrating the second term, we have:

$$\lim_{T\to+\infty}\int_0^T [1+\eta(u)]^2 e^{-2\lambda_{n+1}(T-u)}du = \frac{(1-A)^2}{2\lambda_{n+1}},$$

from which, utilizing the standard results, we have that:

$$\lim_{T\to+\infty}\int_0^T \left[\int_0^s [1+\eta(u)]^2 e^{-2\lambda_{n+1}(s-u)}du\right]ds = +\infty.$$

Utilizing De L'Hopital's rule, we get the thesis.

4 A comparison between the two value functions.

In the previous sections, we have determined the value functions for the two agents: $u(x)$ for the informed one, $v(x)$ for the partially informed one. In this section, we want to focus on the divergence, when $T\to+\infty$,the two utility functions. Besides, when $T\to+\infty$, the difference between the expected utility of two agents, $u(x)$ and $v(x)$, diverges.

We can enunciate the following result:

Theorem 14. *Consider the following properties:*

1.

$$\lim_{T\to+\infty}u(x) = \lim_{T\to+\infty}\left(\log x + \frac{1}{2\sum_{j=1}^{n+1}p_j^2}\int_0^T \frac{\mu_s^2}{\sigma_s^2}ds + \frac{p_{n+1}^2}{\sum_{j=1}^{n+1}p_j^2}\left[\frac{2T\lambda_{n+1}-1+e^{-2T\lambda_{n+1}}}{8}\right]\right) = +\infty$$

2.

$$\lim_{T\to+\infty}v(x) = \lim_{T\to+\infty}\left(\log x + \frac{1}{2\sum_{j=1}^{n+1}p_j^2}\int_0^T \frac{\mu_s^2}{\sigma_s^2}ds + \frac{\lambda_{n+1}^2}{2}\int_0^T\left[\int_0^s[1+\eta(u)]^2e^{-2\lambda_{n+1}(s-u)}du\right]ds\right) = +\infty$$

3.

$$\lim_{T\to+\infty}\frac{u(x)-v(x)}{T} = \frac{\lambda_{n+1}}{2}\left(\sum_{j=1}^{n}p_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n+1}p_j^2\right)^{-1}\left[\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}-\left(\sum_{j=1}^{n}p_j^2\right)^{\frac{1}{2}}\right]$$

as a consequence we have:

$$\lim_{T\to+\infty}[u(x)-v(x)] = +\infty.$$

Proof. The first two properties can be easily verified. About property 3, we note that

$$u(x)-v(x) = \frac{p_{n+1}^2}{\sum_{j=1}^{n+1}p_j^2}\left[\frac{2T\lambda_{n+1}-1+e^{-2T\lambda_{n+1}}}{8}\right] - \frac{\lambda_{n+1}^2}{2}\int_0^T\left[\int_0^s[1+\eta(u)]^2e^{-2\lambda_{n+1}(s-u)}du\right]ds$$

and therefore

$$\frac{u(x)-v(x)}{T}=\frac{p_{n+1}^2}{\sum_{j=1}^{n+1}p_j^2}\left[\frac{2T\lambda_{n+1}-1+e^{-2T\lambda_{n+1}}}{8T}\right]-\frac{\lambda_{n+1}^2}{2T}\int_0^T\left[\int_0^s[1+\eta(u)]^2e^{-2\lambda_{n+1}(s-u)}du\right]ds$$

as a consequence we have

$$\lim_{T\to+\infty}\frac{u(x)-v(x)}{T}=\frac{\lambda_{n+1}}{4}\left(\frac{p_{n+1}^2}{\sum_{j=1}^{n+1}p_j^2}-\left[1-\frac{\left(\sum_{j=1}^{n}p_j^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}}\right]^2\right)$$

it follows that

$$\lim_{T\to+\infty}\frac{u(x)-v(x)}{T}=\frac{\lambda_{n+1}}{4}\left(\frac{p_{n+1}^2}{\sum_{j=1}^{n+1}p_j^2}-1+2\frac{\left(\sum_{j=1}^{n}p_j^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}}-\frac{\sum_{j=1}^{n}p_j^2}{\sum_{j=1}^{n+1}p_j^2}\right)$$

and therefore

$$\lim_{T\to+\infty}\frac{u(x)-v(x)}{T}=\frac{\lambda_{n+1}}{4}\frac{p_{n+1}^2-\sum_{j=1}^{n+1}p_j^2+2\left(\sum_{j=1}^{n}p_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}-\sum_{j=1}^{n}p_j^2}{\sum_{j=1}^{n+1}p_j^2}$$

that is to say

$$\lim_{T\to+\infty}\frac{u(x)-v(x)}{T}=\frac{\lambda_{n+1}}{2}\frac{\left(\sum_{j=1}^{n}p_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}-\sum_{j=1}^{n}p_j^2}{\sum_{j=1}^{n+1}p_j^2}$$

from which, we obtain:

$$\lim_{T\to+\infty}\frac{u(x)-v(x)}{T}=\frac{\lambda_{n+1}}{2}\frac{\left(\sum_{j=1}^{n}p_j^2\right)^{\frac{1}{2}}}{\sum_{j=1}^{n+1}p_j^2}\left[\left(\sum_{j=1}^{n+1}p_j^2\right)^{\frac{1}{2}}-\left(\sum_{j=1}^{n}p_j^2\right)^{\frac{1}{2}}\right]$$

which, simplified, provides the thesis.

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