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The Logarithmic Utility Functions of the Partially Informed Agents

Le Funzioni di Utilitá Logaritmica degli Agenti Parzialmente Informati

Luigi Romano and Donato Scolozzi

Abstract In [7] we extend the asset pricing framework introduced by Guasoni [3] to the case of $n+1$ Brownian motions and two agents: "informed agent" and "partially informed agent". In this paper we examine a model of a financial market with n Brownian motions and four agents: informed agent with access to all information, uninformed agent with access to minimum information, and two partially informed agents with access to partial information that are not necessarily comparable to each other. We examine the information of each agents using Hitsuda representation [4]. In theorem 21 we compare the logarithmic utility functions of the agents. The present framework can be extended to the case with further partially informed agents. For both cases the results for the logarithmic utility functions are similar.

Abstract *In [7] consideriamo una estensione del risultato che Paolo Gausoni ha ottenuto in [3] al caso di $n+1$ moti browniani con due agenti: uno "informato" ed uno "non informato". In questo lavoro esaminiamo un mercato finanziario con n moti browniani e quattro agenti: l'agente informato che conosce tutte le informazioni disponibili, l'agente non informato che conosce il minor numero di informazioni possibili, e due agenti parzialmente informati con informazioni parziali e non necessariamente confrontabili tra loro. Valutiamo per ciascuno di essi le fonti delle rispettive informazioni utilizzando il metodo proposto in [4]. Nel teorema* 21 *confrontiamo le funzioni di utilitá logaritmica degli agenti. L'impostazione fornita puó essere estesa al caso di piú agenti parzialmenti informati con risultati, sulle funzioni di utilitá logaritmiche, simili a quelli ottenuti nel presente lavoro.*

Key words: Stochastic Process, Hitsuda representation, Asymmetric information,

1 Introduction: the model

We consider a financial market with one riskless asset and one risky asset S. The market interest rate is considered deterministic. To describe the dynamics of the risky asset, we consider a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ on which are defined $n = h+k+l+m$ independent Brownian motions, with $n \in \mathbb{N}$, $h \geq 1$, $k \geq 1$, $l \geq 1$, $m \geq 1$:

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$\left(B_t^1\right)_{t\in[0,+\infty[}, \left(B_t^2\right)_{t\in[0,+\infty[}, ..., \left(B_t^h\right)_{t\in[0,+\infty[}$;

$\left(B_t^{h+1}\right)_{t\in[0,+\infty[}, ..., \left(B_t^{h+k}\right)_{t\in[0,+\infty[}$;

$\left(B_t^{h+k+1}\right)_{t\in[0,+\infty[}, ..., \left(B_t^{k+h+l}\right)_{t\in[0,+\infty[}$;

$\left(B_t^{h+k+l+1}\right)_{t\in[0,+\infty[}, ..., \left(B_t^{k+h+l+m}\right)_{t\in[0,+\infty[}$.

If we set the real parameter $\lambda > 0$, we consider the Ornstein-Uhlenbeck process $(U_t)_{t\in[0,+\infty)}$ defined by the following equation

$$U_t + \lambda \int_0^t U_s ds = \sum_{j=1}^m p_{h+k+l+j} B_t^{h+k+l+j}, \quad t \in [0,+\infty[$$

which, as known, is given by:

$$U_t = \sum_{j=1}^m p_{h+k+l+j} \int_0^t e^{-\lambda(t-s)} dB_s^{h+k+l+j}.$$

Then, if we set the real numbers p_j, with $j = 1,2,...,n$, let us consider the process $(Y_t)_{t\in[0,+\infty)}$ defined by:

$$Y_t = \sum_{j=1}^h p_j B_t^j + \sum_{j=1}^k p_{h+j} B_t^{h+j} + \sum_{j=1}^l p_{h+k+j} B_t^{h+k+j} + U_t.$$

Now, let us introduce two deterministic Lebesgue measurable functions

$$\mu, \sigma : [0,+\infty[\longrightarrow [0,+\infty[$$

such that

$$\forall\, T > 0 \qquad \mu \in L^1([0,T]), \quad \sigma \in L^2([0,T]).$$

The price dynamics S of the risky asset evolve according to

$$dS_t = S_t \left[\mu_t dt + \sigma_t dY_t\right]$$

and its solution is:

$$S_t = S_0 \exp\left[\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dY_s\right].$$

In particular, this assumption means that:

1. *we have the "informed agent" with access to all information given by each of the n Brownian motions;*
2. *we have the "uninformed agent" with access to information given by the process Y_t only;*
3. *we have the "first partially informed agent" with access to all information given by the $h+k$ Brownian motions and by the process defined by $A_t = \sum_{j=1}^l p_{h+k+j} B_t^{h+k+j} + U_t$;*

4. *we have the "second partially informed agent" with access to all information given by the $h+l$ Brownian motions and by the process defined by* $A_t^* = \sum_{j=1}^{k} p_{h+j} B_t^{h+j} + U_t$.

The informed agent refers to the filtration $\mathscr{F}^1 = \left(\mathscr{F}_t^1\right)_{t\in[0,+\infty[}$ generated by the n Brownian motions $B_t^1, B_t^2, ..., B_t^n$.

The uninformed agent refers to the filtration $\mathscr{F}^0 = \left(\mathscr{F}_t^0\right)_{t\in[0,+\infty[}$ generated by the process Y_t.

The first partially informed agent refers to the filtration $\mathscr{G} = (\mathscr{G}_t)_{t\in[0,+\infty[}$ generated by the $h+k$ Brownian motions $B_t^1, B_t^2, ..., B_t^h, B_t^{h+1}, B_t^{h+2}, ..., B_t^{h+k}$, and by the process defined by $A_t = \sum_{j=1}^{l} p_{h+k+j} B_t^{h+k+j} + U_t$.

The second partially informed agent refers to the filtration $\mathscr{G}^* = (\mathscr{G}_t^*)_{t\in[0,+\infty[}$ generated by the $h+l$ Brownian motions $B_t^1, B_t^2, ..., B_t^h, B_t^{h+k+1}, B_t^{h+k+2}, ..., B_t^{h+k+l}$, and by the process defined by $A_t^* = \sum_{j=1}^{k} p_{h+j} B_t^{h+j} + U_t$.

All the filtrations considered satisfy the usual conditions of right-continuity and completeness.

Of course, we have that $\mathscr{F}_t^0 \subset \mathscr{G}_t \subset \mathscr{F}_t^1$ and $\mathscr{F}_t^0 \subset \mathscr{G}_t^* \subset \mathscr{F}_t^1$, $\forall\, t > 0$.

We can write the asset price dynamics for the informed agent as:

$$dS_t = S_t \left[(\mu_t - \lambda \sigma_t U_t)\, dt + \sigma_t \sum_{j=1}^{n} p_j dB_t^j \right]$$

which refers to the Brownian motion

$$W_t = \left[\sum_{j=1}^{n} p_j^2 \right]^{-\frac{1}{2}} \sum_{j=1}^{n} p_j B_t^j.$$

The proofs of the theorems in the next sections are similar to those seen in [7].

2 Decomposition of Y_t under $\mathscr{F}^0$

In this section, we'll examine the Markov property of Y_t and will determine, under $\mathscr{F}^0$, the Brownian motion B_t^0 which represents it (see theorem 6). The proofs of this section and the proofs seen in [7] are similar.

Theorem 1. *Y_t is a Gaussian process*

1. $E[Y_t] = 0 \quad \forall t \in [0, +\infty[$.
2. $cov(Y_s, Y_t) = \left(\sum_{j=1}^{h+k+l} p_j^2 \right) t \wedge s + \sum_{j=1}^{m} p_{h+k+l+j}^2 \dfrac{e^{-\lambda|t-s|} - e^{-\lambda(t+s)}}{2\lambda}$.

Let us consider $\lambda > 0$, Y_t is a Markov process if and only if, $p_j = 0, \quad \forall\, j = 1, 2, ..., h+k+l$ or $p_{h+k+l+j} = 0, \ \forall\, j = 1, 2, ..., m$.

In generality Y_t is not a Markov process.

Now we consider the process $(Z_t)_{t\in[0,+\infty[}$ defined by:

$$Z_t = \left(\sum_{j=1}^{n} p_j^2\right)^{-\frac{1}{2}} \left[\sum_{j=1}^{h+k+l} p_j \left(B_t^j + \lambda \int_0^t B_u^j du\right) + \sum_{j=1}^{m} p_{h+k+l+j} B_t^{h+k+l+j}\right].$$

It verifies the following result:

Theorem 2. *Z_t is a Gaussian process*

1. $E[Z_t] = 0 \quad \forall t \in [0,+\infty[$.
2. $cov(Z_t, Z_s) = t \wedge s + \left(\sum_{j=1}^{n} p_j^2\right)^{-1} \left(\sum_{j=1}^{h+k+l} p_j^2\right) \int_0^t \int_0^s \left(\lambda + \lambda^2 u \wedge v\right) du dv.$

Let

$$\tilde{f}(t,s) = -\alpha^2 \left(\lambda + \lambda^2 t \wedge s\right) \quad \forall \ t,s \geq 0$$

where

$$\alpha^2 = \left(\sum_{j=1}^{n} p_j^2\right)^{-1} \sum_{j=1}^{h+k+l} p_j^2.$$

Similarly to Hitsuda [4] and Guasoni [3], in order to derive the Brownian motion B_t^0 we consider the following result.

Theorem 3. *Considering the previous function $\tilde{f}(t,s)$, the following function*

$$\tilde{g}(t,s) = \begin{cases} \lambda \eta(s) & for \quad 0 \leq s \leq t \\ 0 & otherwise \end{cases}$$

verifies the following integral equation

$$\tilde{f}(t,s) = \tilde{g}(t,s) - \int_0^s \tilde{g}(t,u)\, \tilde{g}(s,u)\, du \quad \forall\, 0 \leq s \leq t$$

and $\eta(s)$ verifies the following Cauchy problem

$$\begin{cases} \eta'(s) = \lambda \left[\eta^2(s) - \alpha^2\right] \\ \eta(0) = -\alpha^2. \end{cases}$$

Now we are able to enunciate the following theorem:

Theorem 4. *1. Let $\left(B_t^0\right)_{t\in[0,+\infty[}$ be a Brownian motion under $\left(\mathscr{F}_t^0\right)_{t\in[0,+\infty[}$ such that:*

$$Z_t = B_t^0 - \int_0^t \left(\int_0^s \tilde{g}(s,u)\, dB_u^0\right) ds.$$

2. *Considering the function* $g:[0,T]\times[0,T]\longrightarrow\Re$ *defined by:*

$$g(t,s)=\begin{cases}-\lambda\eta(s)e^{\lambda\int_s^t\eta(u)du} & \text{for } 0\le s\le t\\ 0 & \text{otherwise}\end{cases}$$

we have that:

$$B_t^0=Z_t-\int_0^t\left(\int_0^s g(s,u)\,dZ_u\right)ds.$$

Proof. 1. The existence of the Brownian motion B_t^0 is a consequence of Hitsuda [4] proposition 2, and also of the fact that the function $\tilde{g}(t,s)$ verifies the following integral equation:

$$\tilde{f}(t,s)=\tilde{g}(t,s)-\int_0^s\tilde{g}(t,u)\,\tilde{g}(s,u)\,du\quad\forall\,0\le s\le t.$$

2. We obtain the relation

$$B_t^0=Z_t-\int_0^t\left(\int_0^s g(s,u)\,dZ_u\right)ds$$

in a similar way already seen in [7], theorem 6. $\diamondsuit$

Theorem 5. *Y_t and Z_t verify the following relation:*

$$Y_t+\lambda\int_0^t Y_u du=\left(\sum_{j=1}^n p_j^2\right)^{\frac{1}{2}}Z_t$$

and therefore:

$$Y_t=\left(\sum_{j=1}^n p_j^2\right)^{\frac{1}{2}}\int_0^t e^{-\lambda(t-u)}dZ_u.$$

Proof. $Y_t+\lambda\int_0^t Y_u du=\sum_{j=1}^{h+k+l}p_jB_t^j+U_t+\lambda\left[\int_0^t\left(\sum_{j=1}^{h+k+l}p_jB_u^j+U_u\right)du\right]=$

$$=\sum_{j=1}^{h+k+l}p_jB_t^j+U_t+\lambda\sum_{j=1}^{h+k+l}p_j\int_0^t B_u^j du+\lambda\int_0^t U_u du=\sum_{j=1}^{h+k+l}p_j\left(B_t^j+\lambda\int_0^t B_u^j du\right)+U_t+\lambda\int_0^t U_u du=$$

$$=\sum_{j=1}^{h+k+l}p_j\left(B_t^j+\lambda\int_0^t B_u^j du\right)+\sum_{j=1}^m p_{h+k+l+j}B_t^{h+k+l+j}=\left(\sum_{j=1}^n p_j^2\right)^{\frac{1}{2}}Z_t.\ \diamondsuit$$

Now it is possible to establish the link between the process Y_t and the Brownian motion B_t^0. In particular, we have the following (fundamental) result:

Theorem 6. *Let* $Y_t = \sum_{j=1}^{h+k+l} p_j B_t^j + U_t$ *and* $\left(\mathscr{F}_t^0\right)_{t\in[0,+\infty[}$ *be its completed natural filtration.*

We have that:

1.

$$B_t^0 = \left(\sum_{j=1}^{n} p_j^2\right)^{-\frac{1}{2}} \left[Y_t + \lambda \int_0^t \left(\int_0^s [1+\eta(u)]\, e^{\lambda \int_u^s \eta(q)dq} dY_u\right) ds\right].$$

2.

$$Y_t = \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}} \left[B_t^0 - \lambda \int_0^t \left(\int_0^s [1+\eta(u)]\, e^{-\lambda(s-u)} dB_u^0\right) ds\right].$$

Applying relation 2, we can write the asset price dynamics for the uninformed agent as:

$$dS_t = S_t \left[\left(\mu_t - \lambda \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}} \sigma_t \int_0^t [1+\eta(u)]\, e^{-\lambda(t-u)} dB_u^0\right) dt + \sigma_t \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}} dB_t^0\right].$$

3 Decomposition of A_t under $\mathscr{G}$ and of A_t^* under $\mathscr{G}^*$

In this section, we'll examine the Markov property of

$$A_t = \sum_{j=1}^{l} p_{h+k+j} B_t^{h+k+j} + U_t$$

and of

$$A_t^* = \sum_{j=1}^{k} p_{h+j} B_t^{h+j} + U_t$$

and will determine, under $\mathscr{G}$ and $\mathscr{G}^*$, the Brownian motions C_t^0 e C_t^{0*} which represent them.

Theorem 7. *A_t e A_t^* are Gaussian processes*

1. $E[A_t] = 0, \quad E[A_t^*] = 0 \quad \forall t \in [0,+\infty[$.

2. $cov(A_s, A_t) = \left(\sum_{j=1}^{l} p_{h+k+j}^2\right) t \wedge s + \sum_{j=1}^{m} p_{h+k+l+j}^2 \dfrac{e^{-\lambda|t-s|} - e^{-\lambda(t+s)}}{2\lambda}$;

$cov(A_s^*, A_t^*) = \left(\sum_{j=1}^{k} p_{h+j}^2\right) t \wedge s + \sum_{j=1}^{m} p_{h+k+l+j}^2 \dfrac{e^{-\lambda|t-s|} - e^{-\lambda(t+s)}}{2\lambda}$.

As we have observed for Y_t in theorem 1, also A_t and A_t^*, in generality, are not Markov processes.

Now we consider the processes $(D_t)_{t\in[0,+\infty[}$ and $(D_t^*)_{t\in[0,+\infty[}$ defined by:

$$D_t = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)^{-\frac{1}{2}} \left[\sum_{j=1}^{l} p_{h+k+j}\left(B_t^{h+k+j} + \lambda \int_0^t B_u^{h+k+j} du\right) + \sum_{j=1}^{m} p_{h+k+l+j} B_t^{h+k+l+j}\right];$$

$$D_t^* = \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{-\frac{1}{2}} \left[\sum_{j=1}^{k} p_{h+j}\left(B_t^{h+j} + \lambda \int_0^t B_u^{h+j} du\right) + \sum_{j=1}^{m} p_{h+k+l+j} B_t^{h+k+l+j}\right].$$

They verify the following result:

Theorem 8. *D_t and D_t^* are Gaussian processes.*

1. $E[D_t] = 0 \quad E[D_t^*] = 0 \ \ \forall t \in [0, +\infty[$.
2. $cov(D_t, D_s) = t \wedge s + \left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)^{-1} \left(\sum_{j=1}^{l} p_{h+k+j}^2\right) \int_0^t \int_0^s \left(\lambda + \lambda^2 u \wedge v\right) du dv.$

   $cov(D_t^*, D_s^*) = t \wedge s + \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{-1} \left(\sum_{j=1}^{k} p_{h+j}^2\right) \int_0^t \int_0^s \left(\lambda + \lambda^2 u \wedge v\right) du dv.$

Let

$$\tilde{\varphi}(t,s) = -\beta^2 \left(\lambda + \lambda^2 t \wedge s\right) \quad \forall t, s \geq 0$$

$$\tilde{\varphi}^*(t,s) = -(\beta^*)^2 \left(\lambda + \lambda^2 t \wedge s\right) \quad \forall t, s \geq 0$$

where

$$\beta^2 = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)^{-1} \sum_{j=1}^{l} p_{h+k+j}^2$$

and

$$(\beta^*)^2 = \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{-1} \sum_{j=1}^{k} p_{h+j}^2.$$

Similarly to Hitsuda [4] and Guasoni [3], in order to derive the Brownian motions C_t^0 e C_t^{0*} we consider the following result:

Theorem 9. *1. Considering the previous function $\tilde{\varphi}(t,s)$, the following function*

$$\tilde{\psi}(t,s) = \begin{cases} \lambda \theta(s) & \text{for } 0 \leq s \leq t \\ 0 & \text{otherwise} \end{cases}$$

verifies the following integral equation:

$$\tilde{\varphi}(t,s) = \tilde{\psi}(t,s) - \int_0^s \tilde{\psi}(t,u)\, \tilde{\psi}(s,u)\, du \quad \forall\, 0 \leq s \leq t$$

and $\theta(s)$ verifies the following Cauchy problem

$$\begin{cases} \theta'(s) = \lambda\left[\theta^2(s) - \beta^2\right] \\ \theta(0) = -\beta^2. \end{cases}$$

2. *Considering the previous function $\tilde{\varphi}^*(t,s)$, the following function*

$$\tilde{\psi}^*(t,s) = \begin{cases} \lambda\,\theta^*(s) & \text{for } 0 \le s \le t \\ 0 & \text{otherwise} \end{cases}$$

verifies the following integral equation:

$$\tilde{\varphi}^*(t,s) = \tilde{\psi}^*(t,s) - \int_0^s \tilde{\psi}^*(t,u)\,\tilde{\psi}^*(s,u)\,du \quad \forall\, 0 \le s \le t$$

and $\theta^*(s)$ verifies the following Cauchy problem

$$\begin{cases} \theta^{*\prime}(s) = \lambda\left[\theta^{*2}(s) - (\beta^*)^2\right] \\ \theta^*(0) = -(\beta^*)^2. \end{cases}$$

Now we are able to enunciate the following theorem:

Theorem 10. *1. Let $\left(C_t^0\right)_{t\in[0,+\infty[}$ be a Brownian motion under $(\mathscr{G}_t)_{t\in[0,+\infty[}$ such that:*

$$D_t = C_t^0 - \int_0^t \left(\int_0^s \tilde{\psi}(s,u)\,dC_u^0\right) ds.$$

Considering the function $\psi : [0,T]\times[0,T] \longrightarrow \Re$ defined by:

$$\psi(t,s) = \begin{cases} -\lambda\,\theta(s) e^{\lambda\int_s^t \theta(u)du} & \text{for } 0 \le s \le t \\ 0 & \text{otherwise} \end{cases}$$

we have that:

$$C_t^0 = D_t - \int_0^t \left(\int_0^s \psi(s,u)\,dD_u\right) ds.$$

2. *Let $\left(C_t^{0*}\right)_{t\in[0,+\infty[}$ be a Brownian motion under $(\mathscr{G}_t^*)_{t\in[0,+\infty[}$ such that:*

$$D_t^* = C_t^{0*} - \int_0^t \left(\int_0^s \tilde{\psi}^*(s,u)\,dC_u^{0*}\right) ds.$$

Considering the function $\psi^* : [0,T]\times[0,T] \longrightarrow \Re$ defined by:

$$\psi^*(t,s) = \begin{cases} -\lambda\,\theta^*(s) e^{\lambda\int_s^t \theta^*(u)du} & \text{for } 0 \le s \le t \\ 0 & \text{otherwise} \end{cases}$$

we have that:

$$C_t^{0*} = D_t^* - \int_0^t \left(\int_0^s \psi^*(s,u)\, dD_u^* \right) ds.$$

Theorem 11. *1. A_t and D_t verify the following relation:*

$$A_t + \lambda \int_0^t A_u du = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} D_t$$

and therefore:

$$A_t = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} \int_0^t e^{-\lambda(t-u)} dD_u.$$

2. A_t^* and D_t^* verify the following relation:

$$A_t^* + \lambda \int_0^t A_u^* du = \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2 \right)^{\frac{1}{2}} D_t^*$$

and therefore:

$$A_t^* = \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2 \right)^{\frac{1}{2}} \int_0^t e^{-\lambda(t-u)} dD_u^*.$$

Now it is possible to establish the link between the process A_t and the Brownian motion C_t^0 and between the process A_t^* and the Brownian motion C_t^{0*}. In particular, we have the following (fundamental) result:

Theorem 12. *Let $A_t = \sum_{j=1}^{l} p_{h+k+j} B_t^{h+k+j} + U_t$, $A_t^* = \sum_{j=1}^{k} p_{h+j} B_t^{h+j} + U_t$ and $(\mathscr{G}_t)_{t\in[0,+\infty[}$, $(\mathscr{G}^*{}_t)_{t\in[0,+\infty[}$ be their respective completed natural filtrations.*

We have that:

1.

$$C_t^0 = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{-\frac{1}{2}} \left[A_t + \lambda \int_0^t \left(\int_0^s [1+\theta(u)]\, e^{\lambda \int_u^s \theta(q)dq} dA_u \right) ds \right].$$

2.

$$A_t = \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} \left[C_t^0 - \lambda \int_0^t \left(\int_0^s [1+\theta(u)]\, e^{-\lambda(s-u)} dC_u^0 \right) ds \right].$$

3.

$$C_t^{0*} = \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2 \right)^{-\frac{1}{2}} \left[A_t^* + \lambda \int_0^t \left(\int_0^s [1+\theta^*(u)]\, e^{\lambda \int_u^s \theta^*(q)dq} dA_u^* \right) ds \right].$$

4.

$$A_t^* = \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2 \right)^{\frac{1}{2}} \left[C_t^{0*} - \lambda \int_0^t \left(\int_0^s [1+\theta^*(u)]\, e^{-\lambda(s-u)} dC_u^{0*} \right) ds \right].$$

Applying the previous relation 2, we can write the asset price dynamics for the first partially informed agent as:

$$dS_t = S_t \left[\left(\mu_t - \lambda \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} \sigma_t \int_0^t [1+\theta(u)]\, e^{-\lambda(t-u)} dC_u^0 \right) dt + \sigma_t \left(\sum_{j=1}^{h+k} p_j dB_t^j + \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} dC_t^0 \right) \right]$$

which refers to Brownian motion

$$W_t^1 = \left(\sum_{j=1}^{n} p_j^2 \right)^{-\frac{1}{2}} \left[\sum_{j=1}^{h+k} p_j B_t^j + \left(\sum_{j=1}^{l+m} p_{h+k+j}^2 \right)^{\frac{1}{2}} C_t^0 \right].$$

Applying the previous relation 4, we can write the asset price dynamics for the second partially informed agent as:

$$dS_t = S_t \left[\left(\mu_t - \lambda \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2 \right)^{\frac{1}{2}} \sigma_t \int_0^t [1+\theta^*(u)]\, e^{-\lambda(t-u)} dC_u^{0*} \right) dt + \right.$$
$$\left. + \sigma_t \left(\sum_{j=1}^{h} p_j B_t^j + \sum_{j=1}^{l} p_{h+k+j} B_t^{h+k+j} + \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2 \right)^{\frac{1}{2}} dC_t^{0*} \right) \right]$$

which refers to the Brownian motion

$$W_t^{1*} = \left(\sum_{j=1}^{n} p_j^2 \right)^{-\frac{1}{2}} \left[\sum_{j=1}^{h} p_j B_t^j + \sum_{j=1}^{l} p_{h+k+j} B_t^{h+k+j} + \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2 \right)^{\frac{1}{2}} C_t^{0*} \right].$$

4 The value functions of the agents

Each agent has an initial wealth $x > 0$ and, investing $\forall\ s \in [0,t] \quad t > 0$, H_s units of S_s, he gains the (self-financed) value of the wealth X_t.

Considering the different level of market information of each agents, we can observe that the wealth of the informed agent evolves according to Brownian motion W_t, the wealth of the uninformed agent evolves according to Brownian motion B_t^0, and the wealth of the first partially informed agent and of the second partially informed agent evolves according respectively to Brownian motions W_t^1 and W_t^{1*}.

In this section, we'll examine the utility functions of the four agents.
The (self-financed) value of the wealth X_t at time t is given by:

$$X_t = x + \int_0^t H_s dS_s.$$

For each agents, the process of the units H_t must be predictable in their respective filtrations, $S_t - integrable$ and such that $X_t > 0$ almost certainly and $\forall t \in [0,T]$.

Let $\mathscr{U}$ be the utility function, the agents maximize their expected utility from terminal wealth at time T and they solve the problem:

$$\sup\left\{ E\left[\mathscr{U}\left(x + \int_0^T H_t dS_t\right)\right] : \quad H_t \quad admissible \quad \right\}.$$

To guarantee the positivity of the wealth produced at each time t, we consider the process π_t defined by

$$H_t = \pi_t \frac{X_t}{S_t}.$$

Therefore we obtain

$$X_t = x + \int_0^t \pi_s \frac{X_s}{S_s} dS_s$$

from it we deduce

$$\frac{dX_t}{X_t} = \pi_t \frac{dS_t}{S_t}.$$

Now for the informed agent we have:

$$\frac{dX_t}{X_t} = \pi_t \left(\mu_t - \lambda \sigma_t U_t\right) dt + \pi_t \sigma_t \sum_{j=1}^{n} p_j dB_t^j.$$

For the uninformed agent we have:

$$\frac{dX_t}{X_t} = \kappa_t \left[\mu_t - \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}} \lambda \sigma_t \int_0^t \left[1 + \eta(u)\right] e^{-\lambda(t-u)} dB_u^0\right] dt + \kappa_t \sigma_t \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}} dB_t^0.$$

For the partially informed agents, we have respectively:

$$\frac{dX_t}{X_t} = \kappa_t \left[\mu_t - \lambda \left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)^{\frac{1}{2}} \sigma_t \int_0^t \left[1 + \theta(u)\right] e^{-\lambda(t-u)} dC_u^0\right] dt + \kappa_t \sigma_t \left[\sum_{j=1}^{h+k} p_j dB_t^j + \left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right) dC_t^0\right];$$

$$\frac{dX_t}{X_t} = \kappa_t \left[\mu_t - \lambda \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2 \right)^{\frac{1}{2}} \sigma_t \int_0^t [1+\theta^*(u)]\, e^{-\lambda(t-u)} dC_u^{0*} \right] dt +$$
$$+ \kappa_t \sigma_t \left[\sum_{j=1}^{h} p_j dB_t^j + \sum_{j=1}^{l} p_{h+k+j} dB_t^{h+k+j} + \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2 \right) dC_t^{0*} \right].$$

Now we examine the logarithmic utility functions of the agents.

4.1 The value function of the informed agent

The terminal wealth at time T is given by:

$$X_T = xe^{\int_0^T \left[\pi_s (\mu_s - \lambda \sigma_s U_s) - \frac{1}{2} \left(\sum_{j=1}^{n} p_j^2 \right) \pi_s^2 \sigma_s^2 \right] ds + \left[\sum_{j=1}^{n} p_j^2 \right]^{\frac{1}{2}} \int_0^T \pi_s \sigma_s dW_s}.$$

Theorem 13. *Let $\mathscr{U}(y) = \log y$. The process*

$$\pi_s = \frac{\mu_s - \lambda \sigma_s U_s}{\sum_{j=1}^{n} p_j^2 \sigma_s^2}$$

provides the optimal investment share and the value function is given by:

$$u(x,T) = \log x + \frac{1}{2\sum_{j=1}^{n} p_j^2} E\left[\int_0^T \left(\frac{\mu_s}{\sigma_s} - \lambda U_s \right)^2 ds \right].$$

Proof. $\forall$ π_s admissible, we have that

$$\mathscr{U}(X_T) = \log(X_T) = \log x + \int_0^T \left[\pi_s (\mu_s - \lambda \sigma_s U_s) - \frac{1}{2} \left(\sum_{j=1}^{n} p_j^2 \right) \pi_s^2 \sigma_s^2 \right] ds + \left[\sum_{j=1}^{n} p_j^2 \right]^{\frac{1}{2}} \int_0^T \pi_s \sigma_s dW_s$$

as a consequence, we obtain that:

$$E[\log(X_T)] = \log x + E\left[\int_0^T \left[\pi_s (\mu_s - \lambda \sigma_s U_s) - \frac{1}{2} \left(\sum_{j=1}^{n} p_j^2 \right) \pi_s^2 \sigma_s^2 \right] ds \right].$$

Using standard calculations, we obtain the maximum value.
The optimal portfolio share is given by

$$\pi_s = \frac{\mu_s - \lambda \sigma_s U_s}{\sum_{j=1}^{n} p_j^2 \sigma_s^2}$$

and the maximum value is:

$$u(x,T) = \log x + \frac{1}{2\sum_{j=1}^{n} p_j^2} E\left[\int_0^T \left(\frac{\mu_s}{\sigma_s} - \lambda U_s\right)^2 ds\right]$$

from it we deduce the relation looked for. $\diamondsuit$

Rewriting the previous value function, we have the following result:

Theorem 14. *Let*

$$u(x,T) = \log x + \frac{1}{2\sum_{j=1}^{n} p_j^2} \int_0^T \frac{\mu_s^2}{\sigma_s^2} ds + \frac{\sum_{j=1}^{m} p_{h+k+l+j}^2}{\sum_{j=1}^{n} p_j^2} \left[\frac{T\lambda}{4} - \frac{1-e^{-2T\lambda}}{8}\right].$$

4.2 The value function of the uninformed agent

The terminal wealth at time T is given by:

$$X_T = xe^{\int_0^T \left[\kappa_s \left(\mu_s - \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}} \lambda \sigma_s \int_0^s [1+\eta(u)] e^{-\lambda(s-u)} dB_u^0\right) - \frac{1}{2}\left(\sum_{j=1}^{n} p_j^2\right) \kappa_s^2 \sigma_s^2\right] ds + \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}} \int_0^T \kappa_s \sigma_s dB_s^0}$$

.

Considering the logarithmic utility function we have the following result.

Theorem 15. *Let $\mathscr{V}(y) = \log y$. The process*

$$\kappa_s = \frac{\mu_s - \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}} \lambda \sigma_s \int_0^s [1+\eta(u)] e^{-\lambda(s-u)} dB_u^0}{\left(\sum_{j=1}^{n} p_j^2\right) \sigma_s^2}$$

provides the optimal investment share and the value function is given by:

$$v(x,T) = \log x + \frac{1}{2\left(\sum_{j=1}^{n} p_j^2\right)} E\left[\int_0^T \left(\frac{\mu_s}{\sigma_s} - \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}} \lambda \int_0^s [1+\eta(u)] e^{-\lambda(s-u)} dB_u^0\right)^2 ds\right].$$

Proof. $\forall$ κ_s admissible, we have that

$$\mathscr{V}(X_T) = \log(X_T) = \log x + \int_0^T \left[\kappa_s \left(\mu_s - \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \lambda \sigma_s \int_0^s [1+\eta(u)] e^{-\lambda(s-u)} dB_u^0 \right) - \frac{1}{2} \left(\sum_{j=1}^n p_j^2 \right) \kappa_s^2 \sigma_s^2 \right] ds +$$

$$\left[\sum_{j=1}^n p_j^2 \right]^{\frac{1}{2}} \int_0^T \kappa_s \sigma_s dB_0^s$$

as a consequence, we obtain that

$$E[\log(X_T)] = \log x + E\left[\int_0^T \left[\kappa_s \left(\mu_s - \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \lambda \sigma_s \int_0^s [1+\eta(u)] e^{-\lambda(s-u)} dB_u^0 \right) - \frac{1}{2} \left(\sum_{j=1}^n p_j^2 \right) \kappa_s^2 \sigma_s^2 \right] ds \right].$$

Using standard calculations, we obtain the maximum value.
The optimal portfolio share is given by

$$\kappa_s = \frac{\mu_s - \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \lambda \sigma_s \int_0^s [1+\eta(u)] e^{-\lambda(s-u)} dB_u^0}{\left(\sum_{j=1}^n p_j^2 \right) \sigma_s^2}$$

and the maximum value is:

$$v(x,T) = \log x + \frac{1}{2\sum_{j=1}^n p_j^2} E\left[\int_0^T \left(\frac{\mu_s}{\sigma_s} - \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \lambda \int_0^s [1+\eta(u)] e^{-\lambda(s-u)} dB_u^0 \right)^2 ds \right]$$

from it we deduce the relation looked for. $\diamondsuit$

Rewriting the previous value function, we have the following result:

Theorem 16. *Let*

$$v(x,T) = \log x + \frac{1}{2\sum_{j=1}^n p_j^2} \int_0^T \frac{\mu_s^2}{\sigma_s^2} ds + \frac{\lambda^2}{2} \int_0^T \left[\int_0^s [1+\eta(u)]^2 e^{-2\lambda(s-u)} du \right] ds.$$

To compare the two previous value functions, we consider the following result:

Theorem 17. *Let*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \left[\int_0^s [1+\eta(u)]^2 e^{-2\lambda(s-u)} du \right] ds = \frac{\lambda}{4} (1-\alpha)^2$$

where

$$\alpha = \left(\sum_{j=1}^n p_j^2 \right)^{-\frac{1}{2}} \left(\sum_{j=1}^{h+k+l} p_j^2 \right)^{\frac{1}{2}}.$$

Proof. First, we prove that:

$$\lim_{s\to+\infty}\int_0^s [1+\eta(u)]^2 e^{-2\lambda(s-u)}du = \frac{(1-\alpha)^2}{2\lambda}.$$

Changing variable we obtain that:

$$\int_0^s [1+\eta(s-u)]^2 e^{-2\lambda u}du.$$

Besides, if we denote the indicator function of the interval $[0,s]$ with $I_{[0,s]}$ we can write:

$$\lim_{s\to+\infty}\int_0^{+\infty} [1+\eta(s-u)]^2 e^{-2\lambda u} I_{[0,s]}(u)\,du.$$

Now, note that, since $\lim_{s\to+\infty}\eta(s-u) = -\alpha$, then the integrand function tends punctually to the function $u\in[0,+\infty[\mapsto (1-\alpha)^2 e^{-2\lambda u}$.

Moreover the integrand function is:

$$0 \le [1+\eta(s-u)]^2 e^{-2\lambda u} I_{[0,s]}(u) \le (1+\alpha)e^{-2\lambda u} \quad \forall\, u\in[0,+\infty[$$

and the decrease and increase functions are integrable on $[0,+\infty[$.

From Lebesgue dominated convergence theorem, it follows that:

$$\lim_{s\to+\infty}\left(\int_0^{+\infty} [1+\eta(s-u)]^2 e^{-2\lambda u} I_{[0,s]}(u)\,du\right) = \int_0^{+\infty}(1-\alpha)^2 e^{-2\lambda u}du.$$

Integrating we obtain that:

$$\lim_{s\to+\infty}\int_0^s [1+\eta(u)]^2 e^{-2\lambda(s-u)}du = \frac{(1-\alpha)^2}{2\lambda}$$

from which using standard calculations we have that

$$\lim_{T\to+\infty}\int_0^T\left[\int_0^s [1+\eta(u)]^2 e^{-2\lambda(s-u)}du\right]ds = +\infty.$$

Applying De L'Hopital's rule, we obtain the thesis. $\diamondsuit$

4.3 The value functions of the partially informed agents

The terminal wealth at time T for the partially informed agents is given respectively by:

$$X_T = xe^{\int_0^T\left[\kappa_s\left(\mu_s-\lambda\left(\sum_{j=1}^{l+m}p_{h+k+j}^2\right)^{\frac{1}{2}}\sigma_s\int_0^s[1+\theta(u)]e^{-\lambda(s-u)}dC_u^0\right)\right]ds}$$

$$e^{-\frac{1}{2}\sum_{j=1}^n p_j^2\int_0^T\kappa_s^2\sigma_s^2 ds}\, e^{\int_0^T\kappa_s\sigma_s\left[\sum_{j=1}^{h+k}p_j dB_s^j+\left(\sum_{j=1}^{l+m}p_{h+k+j}^2\right)dC_s^0\right]};$$

$$X_T = xe^{\int_0^T \left[\kappa_s \left(\mu_s - \lambda \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{\frac{1}{2}} \sigma_s \int_0^s [1+\theta^*(u)]\, e^{-\lambda(s-u)} dC_u^{0*}\right)\right] ds}$$

$$e^{-\frac{1}{2}\sum_{j=1}^{n} p_j^2 \int_0^T \kappa_s^2 \sigma_s^2 ds} e^{\int_0^T \kappa_s \sigma_s \left[\sum_{j=1}^{h} p_j dB_t^j + \sum_{j=1}^{l} p_{h+k+j} dB_t^{h+k+j} + \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2\right) dC_s^{0*}\right]}.$$

Considering the logarithmic utility function we have the following result:

Theorem 18. *1. Let $\mathscr{W}(y) = \log y$. The process*

$$\kappa_s = \frac{\mu_s - \lambda \left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)^{\frac{1}{2}} \sigma_s \int_0^s [1+\theta(u)]\, e^{-\lambda(s-u)} dC_u^0}{\left(\sum_{j=1}^{n} p_j^2\right) \sigma_s^2}$$

provides the optimal investment share and the utility function is given by:

$$w(x,T) = \log x + \frac{1}{2\left(\sum_{j=1}^{n} p_j^2\right)} E\left[\int_0^T \left(\frac{\mu_s}{\sigma_s} - \lambda \left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)^{\frac{1}{2}} \int_0^s [1+\theta(u)]\, e^{-\lambda(t-u)} dC_u^0\right)^2 ds\right].$$

2. Let $\mathscr{W}^*(y) = \log y$. The process

$$\kappa_s = \frac{\mu_s - \lambda \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{\frac{1}{2}} \sigma_s \int_0^s [1+\theta^*(u)]\, e^{-\lambda(s-u)} dC_u^{0*}}{\left(\sum_{j=1}^{n} p_j^2\right) \sigma_s^2}$$

provides the optimal investment share and the utility function is given by:

$$w^*(x,T) = \log x + \frac{1}{2\left(\sum_{j=1}^{n} p_j^2\right)} E\left[\int_0^T \left(\frac{\mu_s}{\sigma_s} - \lambda \left(\sum_{j=1}^{k} p_{h+j}^2 + \sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{\frac{1}{2}} \int_0^s [1+\theta^*(u)]\, e^{-\lambda(t-u)} dC_u^{0*}\right)^2 ds\right].$$

Rewriting the previous value functions, we have the following results:

Theorem 19. *1. Let*

$$w(x,T)=\log x+\frac{1}{2\sum\limits_{j=1}^{n}p_j^2}\int_0^T\frac{\mu_s^2}{\sigma_s^2}ds+\frac{\lambda^2}{2}\frac{\sum\limits_{j=1}^{l+m}p_{h+k+j}^2}{\sum\limits_{j=1}^{n}p_j^2}\int_0^T\left[\int_0^s\left[1+\theta(u)\right]^2e^{-2\lambda(s-u)}du\right]ds.$$

2. Let

$$w^*(x,T)=\log x+\frac{1}{2\sum\limits_{j=1}^{n}p_j^2}\int_0^T\frac{\mu_s^2}{\sigma_s^2}ds+\frac{\lambda^2}{2}\frac{\sum\limits_{j=1}^{k}p_{h+j}^2+\sum\limits_{j=1}^{m}p_{h+k+l+j}^2}{\sum\limits_{j=1}^{n}p_j^2}\int_0^T\left[\int_0^s\left[1+\theta^*(u)\right]^2e^{-2\lambda(s-u)}du\right]ds.$$

To compare the previous value functions, we consider the following results:

Theorem 20. *1. Let*

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T\left[\int_0^s\left[1+\theta(u)\right]^2e^{-2\lambda(s-u)}du\right]ds=\frac{1}{2\lambda}(1-\beta)^2$$

where

$$\beta=\left(\sum_{j=1}^{l+m}p_{h+k+j}^2\right)^{-\frac{1}{2}}\left(\sum_{j=1}^{l}p_{h+k+j}^2\right)^{\frac{1}{2}}.$$

2. Let

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T\left[\int_0^s\left[1+\theta^*(u)\right]^2e^{-2\lambda(s-u)}du\right]ds=\frac{1}{2\lambda}(1-\beta^*)^2$$

where

$$\beta^*=\left(\sum_{j=1}^{k}p_{h+j}^2+\sum_{j=1}^{m}p_{h+k+l+j}^2\right)^{-\frac{1}{2}}\left(\sum_{j=1}^{k}p_{h+j}^2\right)^{\frac{1}{2}}.$$

5 A comparison among the value functions of the agents

In the previous sections, we have determinated the value functions of the agents: $u(x,T)$ for the informed agent, $v(x,T)$ for the uninformed agent, $w(x,T)$ for the first partially informed agent and $w^*(x,T)$ for the second partially informed agent. In this section, we want to focus on the divergence, when $T\to+\infty$, the four utility functions. Besides, when $T\to+\infty$, the difference among the expected utility of the four agents as a consequence of the different level of market information of each agents.

Theorem 21. *We have that:*

1.

$$\lim_{T\to+\infty}\frac{u(x,T)-w(x,T)}{T}=\frac{\lambda}{2}\frac{\left(\sum_{j=1}^{l}p_{h+k+j}^2\right)^{\frac{1}{2}}}{\sum_{j=1}^{n}p_j^2}\left[\left(\sum_{j=1}^{l+m}p_{h+k+j}^2\right)^{\frac{1}{2}}-\left(\sum_{j=1}^{l}p_{h+k+j}^2\right)^{\frac{1}{2}}\right]>0;$$

$$\lim_{T\to+\infty}\frac{u(x,T)-w^*(x,T)}{T}=\frac{\lambda}{2}\frac{\left(\sum_{j=1}^{k}p_{h+j}^2\right)^{\frac{1}{2}}}{\sum_{j=1}^{n}p_j^2}\left[\left(\sum_{j=1}^{k}p_{h+j}^2+\sum_{j=1}^{m}p_{h+k+l+j}^2\right)^{\frac{1}{2}}-\left(\sum_{j=1}^{k}p_{h+j}^2\right)^{\frac{1}{2}}\right]>0.$$

2. Considering

$$\alpha=\frac{\left(\sum_{j=1}^{h+k+l}p_j^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n}p_j^2\right)^{\frac{1}{2}}},\quad \beta=\frac{\left(\sum_{j=1}^{l}p_{h+k+j}^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{l+m}p_{h+k+j}^2\right)^{\frac{1}{2}}},\quad \beta^*=\left(\sum_{j=1}^{k}p_{h+j}^2+\sum_{j=1}^{m}p_{h+k+l+j}^2\right)^{-\frac{1}{2}}\left(\sum_{j=1}^{k}p_{h+j}^2\right)^{\frac{1}{2}},$$

we have that

$$\lim_{T\to+\infty}\frac{w(x,T)-v(x,T)}{T}=\frac{\lambda}{4}\left[(1-\beta)^2-(1-\alpha)^2\right]>0$$

$$\lim_{T\to+\infty}\frac{w^*(x,T)-v(x,T)}{T}=\frac{\lambda}{4}\left[(1-\beta^*)^2-(1-\alpha)^2\right]>0;$$

3.

$$\lim_{T\to+\infty}\frac{w(x,T)-w^*(x,T)}{T}=\frac{\lambda}{4}\left[(1-\beta)^2-(1-\beta^*)^2\right].$$

Proof. 1. $$\lim_{T\to+\infty}\frac{u(x,T)-w(x,T)}{T}=\frac{\lambda}{4}\frac{\sum_{j=1}^{m}p_{h+k+l+j}^2}{\sum_{j=1}^{n}p_j^2}-\frac{\lambda}{4}\frac{\sum_{j=1}^{l+m}p_{h+k+j}^2}{\sum_{j=1}^{n}p_j^2}\left[1-\frac{\left(\sum_{j=1}^{l}p_{h+k+j}^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{l+m}p_{h+k+j}^2\right)^{\frac{1}{2}}}\right]^2=$$

$$\frac{\lambda}{4}\frac{1}{\sum_{j=1}^{n}p_j^2}\left[\sum_{j=1}^{m}p_{h+k+l+j}^2-\left[\left(\sum_{j=1}^{l+m}p_{h+k+j}^2\right)^{\frac{1}{2}}-\left(\sum_{j=1}^{l}p_{h+k+j}^2\right)^{\frac{1}{2}}\right]^2\right]=$$

$$\frac{\lambda}{4}\frac{1}{\sum_{j=1}^{n}p_j^2}\left[\sum_{j=1}^{m}p_{h+k+l+j}^2-\sum_{j=1}^{l+m}p_{h+k+j}^2+2\left(\sum_{j=1}^{l+m}p_{h+k+j}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{l}p_{h+k+j}^2\right)^{\frac{1}{2}}-\sum_{j=1}^{l}p_{h+k+j}^2\right]=$$

$$\frac{\lambda}{4}\frac{1}{\sum_{j=1}^{n}p_j^2}\left[2\left(\sum_{j=1}^{l+m}p_{h+k+j}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{l}p_{h+k+j}^2\right)^{\frac{1}{2}}-2\sum_{j=1}^{l}p_{h+k+j}^2\right]=$$

$$\frac{\lambda}{2}\frac{\left(\sum_{j=1}^{l} p_{h+k+j}^2\right)^{\frac{1}{2}}}{\sum_{j=1}^{n} p_j^2}\left[\left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)^{\frac{1}{2}}-\left(\sum_{j=1}^{l} p_{h+k+j}^2\right)^{\frac{1}{2}}\right]>0;$$

$$\lim_{T\to+\infty}\frac{u(x,T)-w^*(x,T)}{T}=\frac{\lambda}{4}\frac{\sum_{j=1}^{m} p_{h+k+l+j}^2}{\sum_{j=1}^{n} p_j^2}-\frac{\lambda}{4}\frac{\sum_{j=1}^{k} p_{h+j}^2+\sum_{j=1}^{m} p_{h+k+l+j}^2}{\sum_{j=1}^{n} p_j^2}(1-\beta^*)^2=$$

$$\frac{\lambda}{4}\left[\frac{\sum_{j=1}^{m} p_{h+k+l+j}^2}{\sum_{j=1}^{n} p_j^2}-\frac{\sum_{j=1}^{k} p_{h+j}^2+\sum_{j=1}^{m} p_{h+k+l+j}^2}{\sum_{j=1}^{n} p_j^2}\left(1-\frac{\left(\sum_{j=1}^{k} p_{h+j}^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{k} p_{h+j}^2+\sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{\frac{1}{2}}}\right)^2\right]=$$

$$\frac{\lambda}{4}\left[\frac{\sum_{j=1}^{m} p_{h+k+l+j}^2}{\sum_{j=1}^{n} p_j^2}-\frac{1}{\sum_{j=1}^{n} p_j^2}\left(\left(\sum_{j=1}^{k} p_{h+j}^2+\sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{\frac{1}{2}}-\left(\sum_{j=1}^{k} p_{h+j}^2\right)^{\frac{1}{2}}\right)^2\right]=$$

$$\frac{\lambda}{4}\frac{1}{\sum_{j=1}^{n} p_j^2}\left[\sum_{j=1}^{m} p_{h+k+l+j}^2-\sum_{j=1}^{k} p_{h+j}^2-\sum_{j=1}^{m} p_{h+k+l+j}^2+\right.$$

$$\left.2\left(\sum_{j=1}^{k} p_{h+j}^2+\sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{k} p_{h+j}^2\right)^{\frac{1}{2}}-\sum_{j=1}^{k} p_{h+j}^2\right]=$$

$$\frac{\lambda}{4}\frac{1}{\sum_{j=1}^{n} p_j^2}\left[2\left(\sum_{j=1}^{k} p_{h+j}^2+\sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{k} p_{h+j}^2\right)^{\frac{1}{2}}-\left(\sum_{j=1}^{k} p_{h+j}^2+\sum_{j=1}^{k} p_{h+j}^2\right)\right]=$$

$$\frac{\lambda}{4}\frac{2\left(\sum_{j=1}^{k} p_{h+j}^2\right)^{\frac{1}{2}}}{\sum_{j=1}^{n} p_j^2}\left[\left(\sum_{j=1}^{k} p_{h+j}^2+\sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{\frac{1}{2}}-\left(\sum_{j=1}^{k} p_{h+j}^2\right)^{\frac{1}{2}}\right]=$$

$$\frac{\lambda}{2}\frac{\left(\sum_{j=1}^{k} p_{h+j}^2\right)^{\frac{1}{2}}}{\sum_{j=1}^{n} p_j^2}\left[\left(\sum_{j=1}^{k} p_{h+j}^2+\sum_{j=1}^{m} p_{h+k+l+j}^2\right)^{\frac{1}{2}}-\left(\sum_{j=1}^{k} p_{h+j}^2\right)^{\frac{1}{2}}\right]>0.$$

2. It is sufficient to prove that $\alpha-\beta>0$:

$$\alpha-\beta=\frac{\left(\sum_{j=1}^{h+k+l} p_j^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}}}-\frac{\left(\sum_{j=1}^{l} p_{h+k+j}^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)^{\frac{1}{2}}}=$$

$$\frac{1}{\frac{\left(\sum_{j=1}^{h+k+l} p_j^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}}}+\frac{\left(\sum_{j=1}^{l} p_{h+k+j}^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)^{\frac{1}{2}}}}\left[\frac{\sum_{j=1}^{h+k+l} p_j^2}{\sum_{j=1}^{n} p_j^2}-\frac{\sum_{j=1}^{l} p_{h+k+j}^2}{\sum_{j=1}^{l+m} p_{h+k+j}^2}\right]=$$

$$\frac{1}{\frac{\left(\sum_{j=1}^{h+k+l} p_j^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}}}+\frac{\left(\sum_{j=1}^{l} p_{h+k+j}^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)^{\frac{1}{2}}}}\frac{\left(\sum_{j=1}^{h+k+l} p_j^2\right)\left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)-\left(\sum_{j=1}^{n} p_j^2\right)\left(\sum_{j=1}^{l} p_{h+k+j}^2\right)}{\left(\sum_{j=1}^{n} p_j^2\right)\left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)}.$$

Considering now the numerator, we have that:

$$\begin{aligned}
&\left(\sum_{j=1}^{h+k+l} p_j^2\right)\left(\sum_{j=1}^{l+m} p_{h+k+j}^2\right)-\left(\sum_{j=1}^{n} p_j^2\right)\left(\sum_{j=1}^{l} p_{h+k+j}^2\right)=\\
&=\left(\sum_{j=1}^{h} p_j^2+\sum_{j=1}^{k} p_{h+j}^2\right)\left(\sum_{j=1}^{k} p_{h+j}^2+\sum_{j=1}^{l} p_{h+k+j}^2\right)+\\
&-\left(\sum_{j=1}^{h} p_j^2+\sum_{j=1}^{k} p_{h+j}^2+\sum_{j=1}^{l} p_{h+k+j}^2\right)\left(\sum_{j=1}^{k} p_{h+j}^2\right)=\\
&=\left(\sum_{j=1}^{h} p_j^2\right)\left(\sum_{j=1}^{k} p_{h+j}^2\right)+\left(\sum_{j=1}^{h} p_j^2\right)\left(\sum_{j=1}^{l} p_{h+k+j}^2\right)+\left(\sum_{j=1}^{k} p_{h+j}^2\right)\left(\sum_{j=1}^{k} p_{h+j}^2\right)+\\
&+\left(\sum_{j=1}^{k} p_{h+j}^2\right)\left(\sum_{j=1}^{l} p_{h+k+j}^2\right)-\left(\sum_{j=1}^{h} p_j^2\right)\left(\sum_{j=1}^{k} p_{h+j}^2\right)-\left(\sum_{j=1}^{k} p_{h+j}^2\right)\left(\sum_{j=1}^{k} p_{h+j}^2\right)+\\
&-\left(\sum_{j=1}^{l} p_{h+k+j}^2\right)\left(\sum_{j=1}^{k} p_{h+j}^2\right)=\left(\sum_{j=1}^{h} p_j^2\right)\left(\sum_{j=1}^{l} p_{h+k+j}^2\right).
\end{aligned}$$

In a similar way we can prove the other limit.

3. The proof is similar to the previous.

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