Rational Theory of Warrant Pricing

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Introduction

This is a compact report on desultory researches stretching over more than a decade.

In connection with stock market fluctuations, L. Bachelier[1], a French mathematician, discovered the mathematical theory of Brownian motion five years before Einstein's classic 1905 paper. Bachelier gave the same formula for the value of a warrant (or "call" or put) based upon this "absolute" or "arithmetic" process that Dr. R. Kruizenga[2] developed years later in a thesis under my direction. Under this formula, the value of a warrant grows proportionally with the square-root of the time to go before elapsing; this is a good approximation to actual pricing of short-lived warrants, but it leads to the anomalous result that a long-lived warrant will increase in price indefinitely, coming even to exceed the price of the common stock itself—even though ownership of the stock is equivalent to a perpetual warrant exercisable at zero price!

The anomaly apparently came because Bachelier had forgotten that stocks possess limited liability and thus cannot become negative, as is implied by the arithmetic Brownian process. To correct this, I introduced the "geometric" or "economic Brownian motion," with the property that every dollar of market value is subject to the same multiplicational or percentage fluctuations per unit time regardless of the absolute price of the stock. This led to the log-normal process for which the value of a call or warrant has these two desired properties: for short times, the $\sqrt{t}$ law holds with good approximation; and for $t \to \infty$, the value of the call approaches the value of the common stock. (All the above assumes that stock-price changes represent a "fair-game" or martingale—or certain trivial generalizations thereof to allow for a fair return. In an unpublished paper and lecture, I made explicit the derivation of this property from the consideration that, if everyone could "know" that a stock would rise in price, it would *already* be bid up in price to make that impossible. See my companion paper appearing in this same issue, entitled "Proof That Properly Anticipated Prices Fluctuate Randomly.")

The above results, which have been presented in lectures since 1953 at M.I.T., Yale, Carnegie, the American Philosophical Society, and elsewhere have also been presented by such writers as Osborne,[3] Sprenkle,[4] Boness,[5] Alexander,[6] and no doubt others.

However, the theory is incomplete and unsatisfactory in the following respects:
1. It assumes, explicitly or implicitly, that the mean rate of return on the warrant is no more than on the common stock itself, despite the fact that the common stock may be paying a dividend and that the warrant may have a different riskiness from the common stock.

° Acknowledgment is made to the Carnegie Corporation for research aid, but sole responsibility for the results is mine.

[1] Ref. [1].

[2] Ref. [2].

[3] Ref. [3].

[4] Ref. [4].

[5] Ref. [5].

[6] Refs. [6], [7].

2. In consequence of the above, the theory implies that warrants (or calls) will never be converted prior to their elapsing date. Necessarily, therefore, no proper theory is provided for the conditions under which warrants will cease to be outstanding.
3. The existing theory, in effect, assumes that the privilege of converting the warrant at any time in the interval (rather than at the end of the period) is worth literally nothing at all.
4. Finally, the theory leads to the mentioned result, that the price of a perpetual warrant should be literally equal to the stock itself—a paradoxical result, and one that does not agree with the observed facts of life (for example, the fact that perpetual Tri-Continental Warrants sell for less than their equivalent amount of common stock, and are in fact being continuously converted into stock in some positive volume).

The present paper publishes, I believe for the first time, the more difficult theory of rationally evaluating a warrant, taking account of the extra worth of the right to convert *at any time in the interval* and deducing the value of the common stock above which it will pay to exercise the warrant. I am glad to acknowledge the valuable contribution of Professor Henry P. McKean, Jr. of the M.I.T. Department of Mathematics, in effecting certain exact solutions and in proving the properties of the general solutions. His analysis appears as a self-contained mathematical appendix. It will be clear that there still remain many unsolved problems. (For example, *exact* explicit solutions are now known in the case of perpetual warrants only for three cases: the log-normal, the log-Poisson, and the case where the only two possibilities are those of instantaneous complete loss or of a gain growing exponentially in time. Only for this last case is an exact explicit solution known for the finite-time warrant. These exact solutions, which are all due to McKean, correspond to various intuitive conjectures and empirical patterns and can be approximated by the solutions to the simpler problem of discrete, albeit small, time periods.)

The Postulated Model

Let the price of a particular common stock be defined for all time and be denoted by X_t. If we stand at the time t, we know with certainty X_t (and all of its past values X_{t-T}). Its future price X_{t+T} is knowable only in some probability sense, its probability distribution being in the most general case a function of the whole past profile of X_{t-T}. A special simplification involves postulating a Markov property to the process, so that future X_{t+T} has a distribution depending only on present X_t—namely

$$(1)\quad \text{Prob}\{X_{t+T} \leqq X \mid X_t = x\} = P(X,x;T).$$

Obviously, (1) involves the critical assumption of a "stationary time series."

I further posit that each dollar of present value must be expected to have some mean gain per unit time, α, where α may perhaps be zero or more likely will be a positive quantity whose magnitude depends on the dispersion riskiness of X_t and the typical investor's utility aversion to risk. (A deeper theory would posit concave utility and deduce the value of α for each category of stocks.) This expected-returns axiom says

$$(2)\quad E[X_{t+T} \mid X_t] = \int_0^\infty X dP(X,X_t;T) = X_t e^{\alpha t}, \ \alpha \geqq 0$$

(Since money bears the safe return of zero, α cannot be less than zero for risk averters; indeed, it cannot be less than the safe return or pure interest on funds, if such exists. If utility were convex rather than concave, people might be willing to pay for riskiness, and α might be permitted to be negative—but not here.)

The integral in (2) is the usual Stieltjes integral: if the probability distribution $P(X,X_t;T)$ has a regular probability density $\partial P(X,X_t;T)/\partial X = p(X,X_t;T)$, we have the usual Riemann integral $\int_0^\infty Xp(X,X_t;T)dX$; if only discrete probabilities are involved, at $X = X_i$ with probabilities $P_i(X_t;T)$, the integral of (2) becomes the sum $\Sigma X_i P_i(X_t;T)$, which may involve a finite or countably-infinite number of terms. The reader can use the modern notation $\int_0^\infty XP(dX,X_t;T)$ rather than that of (2) if he prefers.

In (2) the limit of integration is given as 0 rather than $-\infty$, because of the important phenomenon of limited liability. A man cannot lose more than his original investment: General Motors stock can drop to zero, but not below.

If the probability of a future price X_{t+T} depends solely on knowledge of X_t alone, having the Markov property of being independent of further knowledge of past prices such as $X_{t-\Delta}$, then

$$(3) \quad P(X_{t+T}|X_t,X_{t-\Delta}) \equiv P(X_{t+T}|X_t)$$

and (1) will satisfy the so-called Chapman-Kolmogorov equation

$$(4) \quad P(X_{t+T},X_t;T) \equiv \int_0^\infty P(X_{t+T},x;T-S)\,dP(x,X_t;S),\ 0 \leq S \leq T.$$

Remarks about Alternative Axioms

To see the meaning of this, suppose t takes on only discrete integral values. Then, without the Markov property (3), (1) would have the general form

$$(5) \quad \text{Prob}\{X_{t+k} \leq X|X_t,X_{t-1},\ldots\} = P(X,X_t,X_{t-1},\ldots;k)$$

with

$$(2)' \quad E[X_{t+1}|X_t,X_{t-1},\ldots] = \int_0^\infty X\,dP(X,X_t,X_{t-1},\ldots;1) = e^{\alpha}X_t.$$

Instead of (4), we would have

$$(4)' \quad P(X_{t+2},X_t,X_{t-1},\ldots;2) = \int_0^\infty P(X_{t+2},X_{t+1},X_t,\ldots;1)\,dP(X_{t+1},X_t,X_{t-1},\ldots;1)$$

where the integration is over X_{t+1} and where X_t is seen to enter in the first factor of the integrand. Even without the Markov axiom of (3), from (2)' applied to the next period's gains, we could deduce the truth of (2) for two periods' gains as well and, by induction, for all-periods' gains—namely

$$E[X_{t+2}|X_t,X_{t-1},\ldots]$$
$$= \int_0^\infty X\,dP(X,X_t,X_{t-1},\ldots;2)$$
$$= \int_0^\infty X\,d\int_0^\infty P(X,x,X_t,X_{t-1},\ldots;1)\,dP(x,X_t,X_{t-1},\ldots;1)$$
$$(6)$$
$$= \int_0^\infty e^{\alpha}x\,dP(x,X_t,X_{t-1},\ldots;1)$$
$$= e^{2\alpha}X_t.$$

Then, by induction, (2) or

$$E[X_{t+k}|X_t,X_{t-1},\ldots] = e^{k\alpha}X_t$$

follows from the weak assumption of (5) and (2)' alone even when the Markov property (3) and Chapman-Kolmogorov property (4) do not necessarily hold.

However, I shall assume (3), and *a fortiori* (4), in order that the rational price of a warrant be a function of current common stock price X_t alone and not be (at this level of approximation) a functional of all past values X_{t-T}. A more elaborate theory would introduce such past values, if only to take account of the fact that the numerical value of α will presumably depend upon the estimate from past data that risk averters make of the riskiness they are getting into when holding the stock.

I might finally note that Bachelier assumed implicitly or explicitly

$$(7) \quad P(X,x;T) \equiv P(X-x;T), \quad \alpha = 0$$

so that an absolute Brownian motion or random walk was involved. He thought that he could deduce from these assumptions alone the familiar Gaussian distribution—or, as we would say since 1923, a Wiener process—but his lack of rigor prevented him from seeing that his form of (4):

$$(8) \quad P(X-x;T) \equiv \int_{-\infty}^{\infty} P(X-x-y;T-S)\,dP(y;S),\ 0 \leq S \leq T$$

does have for solutions, along with the Gaussian distribution, all the other members of the Lévy-Khintchin family of infinitely-divisible distributions[7], including the stable distribution of Lévy-Pareto, the Poisson distribution, and various combinations of Poisson distributions.

The "Geometric or Relative Economic Brownian Motion"

As mentioned, Bachelier's absolute Brownian motion of (7) leads to negative values for X_{t+T} with strong probabilities. Hence, a better hypothesis for an economic model than $P(X,x;T) = P(X - x;T)$ is the following

$$(9) \quad P(X,x;T) \equiv P\left(\frac{X}{x};T\right), \quad x > 0$$

$$P(X,0;T) \equiv 1 \text{ for all } X > 0.$$

[7] Refs. [8], [9].

By working with ratios instead of algebraic differences, we consider logarithmic or percentage changes to be subject to uniform probabilities. This means that the first differences of the logarithms of prices are distributed in the usual absolute Brownian way. Since the arithmetic mean of logs is the geometric mean of actual prices, this modified random walk can be called the geometric Brownian motion in contrast to the absolute or arithmetic Brownian motion.

The log-normal distribution bears to the geometric Brownian motion the same relation that the normal distribution does to the ordinary Brownian motion. As the writings of Mandelbrot[8] and Fama[9] remind us, there are non-log-normal stable Pareto-Lévy distributions (of logs) satisfying the following form of (4):

$$(10)\quad P\left(\frac{X}{x};T\right) \equiv \int_0^\infty P\left(\frac{X}{y};T\text{-}S\right) dP\left(\frac{y}{x},S\right).$$

Some of our general results require only that (1), (2), and (4) hold. But most of our explicit solutions are for multiplicative processes, in which (9), (10) and the following hold:

$$(11)\quad E[X_{t+T}|X_t] = \int_0^\infty X dP\left(\frac{X}{X_t};T\right) = X_t e^{\alpha T}, \quad \alpha \geqq 0.$$

Actually, (9) and (10) alone require that the family $P(X;T)$ is determined once a single admissible function $P(X;T_1) = P(X)$ is given, as for $T_1 = 1$. Then if α is defined by

$$(12)\quad e^{\alpha T_1} = E[X_{t+T_1}/X_t] = \int_0^\infty X dP(X),$$

(11) is provable as a theorem and need not be posited as an axiom. McKean's appendix assumes the truth of (9) and (10) throughout. It is known from the theory of infinitely-divisible processes that $P(X)$ above cannot be an arbitrary distribution but must have the characteristic function for its log, $Y = \log X$, of the Lévy-Khintchin form:

$$(13)\quad E[e^{i\lambda Y}] = \int_{-\infty}^{\infty} e^{i\lambda Y} dP(e^Y) = e^{g(\lambda)}$$

$$g(\lambda) = \mu i\lambda + \int \left(e^{i\lambda z} - 1 - \frac{i\lambda z}{1+z^2} \right) \frac{1+z^2}{z^2} d\psi(z),$$

where $\psi(z)$ is itself a distribution function. In the special cases of the log-normal distribution, the log-Poisson distribution, and the log-Lévy distribution, we have respectively

$$g(\lambda) = \mu i\lambda - \frac{\sigma^2}{2}\lambda^2$$

$$(14)\quad g(\lambda) = e^{i\lambda\mu} - 1$$

$$g(\lambda) = \mu i\lambda - \gamma|\lambda|^{\alpha^*}[1 + i\beta(\lambda/|\lambda|)\tan(\alpha^\circ\pi/2)],\ 0 \leqq \alpha^\circ \leqq 2.$$

All of (14) is on the assumption that

$$\lim_{X\to 0} P(X) = P(0) = 0.$$

If $P(0) > 0$, there is a finite probability of complete ruin in any time interval, and as the interval approaches infinity that probability approaches 1. An example (the only one for which *exact* formulas for rational warrant pricing of all durations are known) is given by

$$(15)\quad \begin{aligned} &\text{Prob}\{X_{t+T} = X_t e^{aT}\} = e^{-bT} \quad a,b > 0 \\ &\text{Prob}\{X_{t+T} = 0\} = 1 - e^{-bT} \end{aligned}$$

where $\alpha = a - b \geqq 0$.

Letting $w(X;T)$ be an infinitely-divisible (multiplicative) function satisfying (13), the most general pattern would be one where

$$P(0,T) = 1 - e^{-bT} \qquad b > 0$$

$$(16)\quad P(X,T) = e^{-bT} w(X,t) + P(0,T)$$

$$\text{with } P(\infty,T) = e^{-bT}1 + 1 - e^{-bT} = 1.$$

One final remark. Osborne, by an obscure argument that appeals to Weber-Fechner and to clearing-of-free-markets reasoning, purports to deduce, or make plausible, the axiom that the geometric mean of the distribution $P(X/x;T)$ is to be unity or that the expected value of the logarithmic difference is to be a random walk without mean bias or drift. Actually,

[8]Ref. [10].

[9]Ref. [11].

if $\alpha = 0$ in (2), so that absolute price is an unbiased martingale, the logarithmic difference must have a negative drift. For α sufficiently positive, and depending on the dispersion of the log-normal process, the logarithmic difference can have any algebraic sign for its mean bias. Only if one could be sure that $P(X/X;T) = P(1;T) \equiv \frac{1}{2}$, so that the chance of a rise in price could be known to be always the same as the chance of a fall in price, would the gratuitous Osborne condition turn out to be true.

If $P(X, x; T)$ corresponds to a martingale or "fair game," with $\alpha = 0$ as in the Bachelier case, the arithmetic mean of the ratio X/x is always exactly 1 and the geometric mean, being less than the arithmetic mean if P has any dispersion at all, is less than 1. Its logarithm, the mean or expected value of $\log X_{t,T}/X_t$ is then negative, and the whole drift of probability for $P(X,x;T)$ shifts leftward or downward through time. In long enough time, the probability approaches certainty that the investor will be left with less than 1 cent of net worth—i.e., $P(0+,x;\infty) = 1$. This virtual certainty of almost-complete ruin bothers many writers. They forget, or are not consoled by, the fact that the gains of those (increasingly few) people who are not ruined grow prodigiously large—in order to balance the complete ruin of the many losers. Therefore, many writers are tempted by Osborne's condition, which makes the expected median of price $X_{t,T}$ neither grow above nor decline below X_t.

However, in terms of present discounted value of future price, $X_{t,T}e^{-\alpha T}$, where the mean yield α is used as the discount factor, most people's net worth does go to zero, and this occurs *in every case of* $\alpha \geqq 0$. Relative to the expected growth of $X_{t,T}$—i.e., relative to $X_t e^{\alpha T}$, $X_{t,T}$ does become negligible with great probability. I call this condition "relative ruin," with the warning that a man may be comfortably off and still be ruined in this sense. And I now state the following general theorem:

Theorem of virtual certainty of (relative) ruin: Let $P(X,x;T)$ have non-zero dispersion, satisfying

$$\int_0^\infty X\, d\, P(X,x;T) \equiv e^{\alpha T},$$

$$P(X,x;T) = \int_0^\infty P(X,y;\, T-S)dP(y,x;S),\ \alpha \geqq 0$$

as in (2) and (4). Then

$$\lim_{T\to\infty} P(X\, e^{-\alpha T}, x;T) = 1 \qquad \text{for all } (X,x) > 0$$

In the multiplicative-process case, $P(X, x; T) = P(X/x; T)$ and the theorem follows almost directly from the fact that the geometric mean is less than the arithmetic mean.

In words, the theorem says that, with the passage of ever longer time, it becomes more and more certain that the stock will be at a level whose present discounted value (discounted at the expected yield α of the stock) will be less than 1 cent, or one-trillionth of a cent.

As is discussed on page 30, we can replace relative ruin by absolute ruin whenever the dispersion of the log-normal process becomes sufficiently large. Thus, even if $\alpha > 0$ in accordance with positive expected yield, whenever the parameter of dispersion $\sigma^2 > 2\alpha$, there is virtual certainty of absolute ruin. Indeed, for the log-normal case we can sharpen the theorem to read

$$\lim_{T\to\infty} P(0+, x; T) = 1, \quad \sigma^2 > 2\alpha$$

Summary of Probability Model

The $X_{t,T}$ price of the common stock is assumed to follow a probability distribution dependent in Markov fashion on its X_t price alone and on the elapsed time:

$$(1)\quad \text{Prob}\{X_{t,T} \leq X|X_t\} = P(X,X_t;T)$$

$$(4)\quad P(X_{t,T},X_t;T) \equiv \int_0^\infty P(X_{t,T},x;T\text{-}S)dP(x,X_t;S),\ 0 \leq S \leq T$$

with the expected value of price assumed to have a constant mean percentage growth per unit time of α, or

$$(2)\quad E[X_{t,T}|X_t] = X_t e^{\alpha T} = \int_0^\infty X dP(X,X_t;T),\quad \alpha \geqq 0.$$

In many cases $P(X,x;T)$ will be assumed to be a multiplicative process, with the ratio X_{t+T}/X_t independent of all X_{t-W}. Then we can write

$$P(X,x;T) \equiv P\left(\frac{X}{x};T\right),$$

where $P(u;T)$ belongs to the special family of infinitely-divisible (multiplicative) distributions of which the log-normal, log-Poisson, and log-Lévy functions are special cases. (If the Lévy coefficient α° in (14), which must not be confused with α of (2), were not 2 as in the log-normal case, we can show that α in (2) will be infinite. Ruling out that case will rule out the Lévy-Pareto-Mandelbrot distributions.)

Arbitrage Conditions on Warrant Prices

A warrant is a contract that permits one to buy one share of a given common stock at some stipulated exercise price X° (here assumed to be unchangeable through time, unlike certain real-life changing-terms contracts) at any time during the warrant's remaining length of life of T time periods. Thus, a warrant to buy Kelly, Douglas stock at \$4.75 per share until November, 1965, has $X^\circ = \$4.75$ and (in March, 1965) has $T = 7/12$ years. A perpetual warrant to buy Allegheny Corporation at \$3.75 per share has $X^\circ = \$3.75$ and $T = \infty$.

When a warrant is about to expire and its $T = 0$, its value is only its actual conversion value. If the stock now has $X_t = X^\circ$, with the common selling at the exercise price to anyone whether or not he has a warrant, the warrant is of no value. If $X_t < X^\circ$, *a fortiori* it is worth nothing to have the privilege of buying the stock at *more* than current market price, and the warrant is again worthless. Only if $X_t > X^\circ$ is the expiring warrant of any value, and—brokerage charges being always ignored—it is then worth the positive difference $X_t - X^\circ$.

In short, arbitrage alone gives the rational price of an expiring warrant with $T = 0$, as the following function of the common price known to be $X_t = X$, $F(X,T) = F(X,0)$, where

$F(X,0) = \mathrm{Max}[0, X - X^\circ]$.

A warrant good for $T_1 > 0$ periods is worth at least as much as one good only for $T_2 < T_1$ periods and generally is worth more. Hence, arbitrage will ensure that the rational price for a warrant with T_1 time to go, denoted by $F(X,T_1)$, will satisfy

$$F(X,T_1) \geqq F(X,T_2) \quad \text{if } T_1 \geqq T_2.$$

A perpetual warrant is one for which $T = \infty$. But recall that outright ownership of the common stock, aside from giving the owner any dividends the stock declares, is equivalent to having a perpetual warrant to buy the stock (from himself!) at a zero exercise price. Hence, a perpetual warrant cannot now sell for more than the current price of the common stock Or, in general, arbitrage requires that

$$(17) \quad X \geqq F(X,\infty) \geqq F(X,T_1) \geqq F(X,T_2) \geqq F(X,0) = \mathrm{Max}[0,X-X^\circ]$$

where

$$\infty \geqq T_1 \geqq T_2 \geqq 0.$$

In all that follows we shall, by an admissible choice of conventional units, be able to assume that the exercise price is $X^\circ = 1$. Thus, instead of working with the price of one actual Kelly, Douglas warrant, which gives the right to buy one share of Kelly, Douglas common stock at \$4.75, we work with the standardized variable $X/X^\circ = X/4.75$—the number of shares purchasable at \$1, which is of course 1/4.75 actual shares; correspondingly, the warrant price Y_t we work with is not the actual Y_t but the standardized variable Y_t/X°, which represents the price of a warrant that enables the holder to buy 1/4.75 actual shares at the exercise price of \$1. We are able to do this by the following homogeneity property of competitive arbitrage:

$$(18) \quad \frac{F(X,X^\circ;T)}{X^\circ} \equiv F\left(\frac{X}{X^\circ},1;T\right),$$

a property that says no more than that two shares always cost just twice one share. Wherever we write $F(X;T)$, we shall really be meaning (18). (Note that Tri-Continental perpetual warrants involve the right to buy 1.27 shares at \$17.76 per share. In calculating $X/X^\circ = X/17.76$, we use for X the price of 1.27 shares, not of one share.)

Our conventions with respect to units ought to be adopted by advisory services dealing with warrants, to spare the reader the need to calculate X/X° and Y/Y°. All this being understood, we can rewrite the fundamental inequalities of arbitrage shown in (17) as follows:

$$(19)\quad X \geqq F(X,\infty) \geqq F(X,T_1) \geqq F(X,T_2) \geqq F(X,0) = \text{Max}(0,X-1), \infty \geqq T_1 \geqq T_2 \geqq 0.$$

In Figures 1a and 1b, the outer limits are shown in heavy black: OAB is the familiar function Max(0,X-1). (In McKean's appendix, this is written in the notation $(X-1)^{+}$.) The 45° line OZ represents the locus whose warrant price equals X, the price of the common stock itself.

Axiom of Expected Warrant Gain

Mere arbitrage can take us no further than (19). The rest must be experience—the recorded facts of life. Figure 1a shows one possible pattern of warrant pricing. The expiring warrant, with $T = 0$, must be on the locus given by OAB. If positive length of life remains, $T > 0$, Figure 1a shows the warrant always to be worth more than its exercise price: thus, OCD lies above OAB for all positive X; because OEF has four times the length of life of OCD, its value at $X = 1$ is about twice as great—in accordance with the rule-of-thumb $\sqrt{T}$ approximation; because T is assumed small, and $P(X/x;T)$ approximately symmetrical around $X/x = 1$, the slope at C is about ½—in accordance with the rule-of-thumb approximation that if two warrants differ only in their exercise price X°, the owner should pay \$1/2 for each \$1 reduction in X°, this being justifiable by the reasoning that there is only a half-chance that he will end up exercising at all and benefitting by the X° reduction. Note that all the curves in the figures are convex (from below) and all but the OAB and OZ limits are drawn to be strictly convex (as would be the case if $P(u;1)$ were log-normal or a distribution with continuous probability density). Our task is to demonstrate rigorously that the functions shown in the figures are indeed the only possible rational pricing patterns.

The pricing of a warrant becomes definite once we know the probability distribution of its common stock $P(X,x;T)$ if we pin down buyers' reactions to the implied probability distribution for the warrant's price $Y_t(T_t)$, in the form of the following axiom:

Axiom of mean expectation. Whereas the common stock is priced so that its mean expected percentage growth rate per unit time is a non-negative constant α, the warrant is priced so that it, too, will have a constant mean expected percentage growth rate per unit time for as long as it pays to hold it, the value of the constant being at least as great as that for the stock—or $\beta \geqq \alpha$. Mathematically

$$(20)\quad E[Y_{t+T}(T_t - T) \mid Y_t(T_t)] = e^{\beta T}Y(T_t)$$

for all times T it pays to hold the warrant, where

$$(21)\quad \beta \geqq \alpha = \log_e \int_0^{\infty} X dP(X, x; 1) \geqq 0.$$

The reader should be warned that the expected value for the warrant in (20) is more complicated than the expected value of the stock in (2). The latter holds for any prescribed time period; but in (20), the time period T must be one in which it pays to have the warrant held rather than converted. (In the appendix, McKean's corresponding expectation is given in 2.8 and in 4.8.) It is precisely when the warrant has risen so high in price (above C_T in Figure 1b) that it can no longer earn a stipulated positive excess β-α over the stock that it *has* to be converted. Actually if β is stipulated to equal α, we are in Figure 1a rather than Figure 1b: there is never a need to convert before the end of life, and hence all points like $C_0, \ldots C_T$ are at infinity; as we shall see, the conventional linear integral equations enable us easily to compute the resulting functions in Figure 1a.

Warrants, unlike calls, are not protected against the payment of dividends by the common stock. Hence, for any stock that pays a positive dividend, say at the instantaneous rate of δ times its market value, the warrant will have to have a $\beta > \alpha$ if it is to represent as good a buy as the stock itself. Taxes and peculiar subjective reactions to the riskiness patterns of the two securities aside, at the least $\beta=\alpha+\delta>\alpha$. However, even if $\delta = 0$ and there is no dividend, buyers may feel that the volatility pattern of warrants is such that owners must be paid a greater mean return to hold warrants than to hold stocks. I do not pretend to give a theory from which one

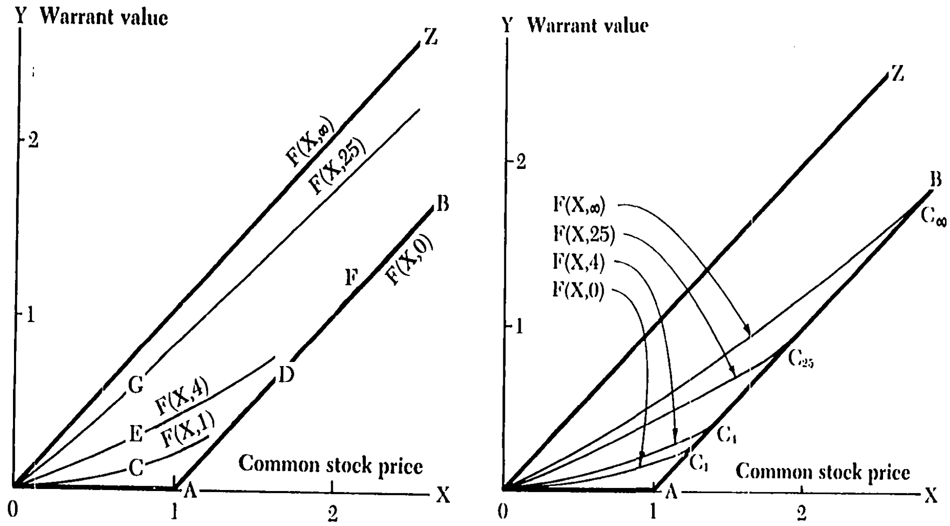

Figure 1a *Figure 1b*

Figure 1 Rational Warrant Pricing

These graphs show the general pattern of warrant pricing as a function of the common stock price (where units have been standardized to make the exercise price unity). The longer the warrant's life T, the higher is F(X,T). For fixed T, F(X,T) is a convex function of X. In Figure 1a, the perpetual warrant's price is equal to that of the stock, with $F(X,\infty)$ falling on OZ; it never pays to exercise such a warrant. In Figure 1b, the points C_1, C_4, C_{25}, and C_∞ on AB are the points at which it pays to convert a warrant with $T = 1$, 4, 25 and ∞ years to run. Note that $F(X,\infty)$ is much less than X in this case. The pattern of Figure 1b will later be shown to result from the hypothesis that a warrant must have a mean yield β greater than the stock's mean yield α.

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can deduce the relative values of β and α. Here, I merely postulate that they are constants (independent, incidentally, of T, the life span of the warrant).

My whole theory rests on the axiomatic hypotheses:

1. The stock price is a definite probability distribution, $P(X,x;T)$, with constant mean expected growth per unit time $\alpha \geqq 0$.
2. The warrant's price, derivable from the stock price, must earn a constant mean expected growth per unit time, $\beta \geqq \alpha \geqq 0$.

Once these axioms, the numbers α,β, and the form of $P(X,x;T)$ are given, it becomes a determinate mathematical problem to work out the rational warrant price functions $Y_t(T_t) = F(X_t,T_t)$ for all non-negative T_t, including the perpetual warrant $F(X_t,\infty)$.

Some Intuitive Demonstrations

Before giving the mathematical solutions, I shall indicate how one can deduce the paradoxical result that a perpetual warrant must have the same price as the common stock if they both have to earn the same mean yield. The reader may want to think of the fair-game case where $\beta = \alpha = 0$, a case which has a disproportionate fascination for economists because they wrongly think that if prices were known to be biased toward rising in the future, that fact would already be "discounted" and the price would already have risen to the point where α can be expected to be zero. (What is forgotten here by Bachelier and others—but not by Keynes, Houthakker, Cootner[10], and other exponents of "normal backwardation"—is that time may involve money, opportunity cost, and risk aversion.)

A warrant is said to involve "leverage" in comparison with the common stock, and in the real world where brokerage charges

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[10] Refs. [12], [13], [14], [15], [16]. See my cited companion paper in this issue.

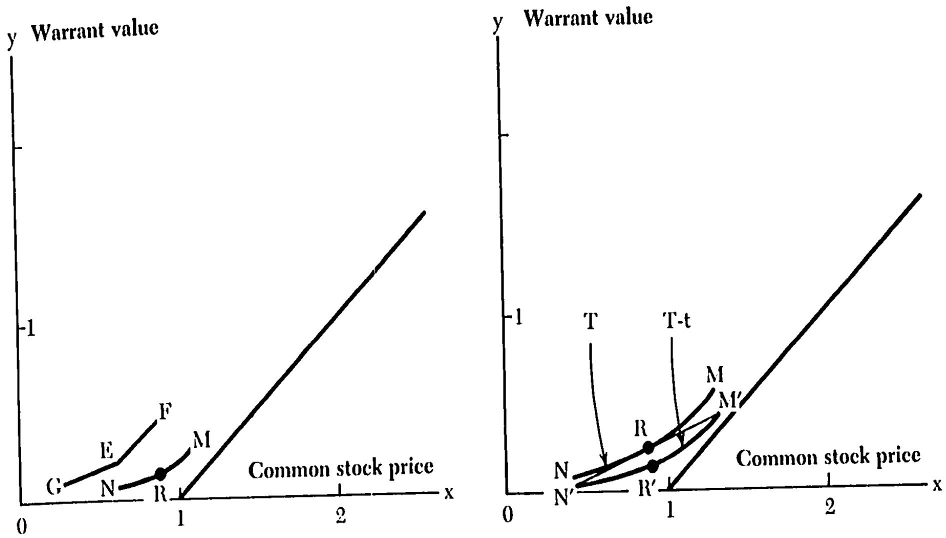

Figure 2 **Warrant Pricing—The Perpetual Case**

and imperfect capital rationing are involved, leverage can make a difference. The exact meaning of leverage is not always clear, and writers use the term in two distinct senses. The usual sense is merely one of *percentage* volatility. Suppose a stock is equally likely to go from $10 to $11 or to $9. Suppose its warrant is equally likely to go from $5 to $6 or to $4. Both are subject to a $1 swing in either direction; but $1 on $5 is twice the percentage swing of $1 on $10, as will be seen if equal dollars were invested in each security. In this sense, the warrant would be said to have twice the leverage of the stock. Leverage in the sense of mere enhanced percentage variability is a two-edged sword: as much as it works for you on the upside, it works against you on the downside. It is perfectly compatible with $\alpha = \beta = 0$. (However, if there were a two-thirds chance of each security's going up $1 and a one-third chance of its going down $1, the warrant's β would be definitely greater than stock's α, since a mean expected return of +33 1/3¢ on $5 is twice that of +33 1/3¢ on $10; and this impinges on the second sense of leverage.)

The second sense of the term leverage is merely enhanced expected yield from the warrant in comparison with the common stock. Here is an example from the R.H.M. Warrant and Stock Survey of February 25, 1965: Newconex Holdings Warrant, Toronto Exchange, "would rise about 2.25 times as fast as the common stock on the upside and decline no faster than the common on the downside." To one who believes this, the warrant offers very good value or "leverage" in this second sense of the term. Indeed, by selling one common short and buying one warrant, one could presumably break even if the stock went down in price and make money if the stock rises—a sure-thing hedge that cannot lose if one believes the stated probability judgment.

Figure 2 shows for a hypothetical perpetual warrant a convex corner at the existing price E, with EF steeper for a rise than EG for a fall. Obviously, GEF could not persist if the warrant's β gain were to be no bigger than the stock's α gain. Similarly, the strongly convex NRM could not persist with $\beta = \alpha$. What pattern for a perpetual warrant could persist? Only a straight-line pattern, since for any convexity at all the mean of points along a curve must lie above the curve itself.

What straight line can be fitted in between OZ and OAB of Figure 1? Obviously, only the line OZ itself—proving that the only rational price for a perpetual warrant must be that of the common stock itself when $\alpha = \beta$. (Any straight line not parallel to OZ and AB will intersect one or both of them; any intermediate line parallel to OZ and AB will hit the zero axis at positive X and then develop a corner there. So OZ alone remains as the formula for $F(X,\infty) \equiv X$.)

The curve of $F(X,T)$ for finite T can and will be convex. But as time passes, one does not move up and down the curve itself—say from R to M if X rises or from R to N if X falls. Instead, as time passes T diminishes, and one moves from R to a point below M or N on the new $F(X,T\text{-}t)$ curve; and if the two convex curves have been sketched correctly and placed in the proper shift relationship to each other, it will be found that the mean expectation of gain from the warrant is precisely that from the stock.

The moral of this is not that surveys are wrong when they recommend a bargain. It is rather that one recognizes correct or rational pricing and the absence of bargains when the warrants are priced in a certain way relative to the common stock. It is only as people act to take advantage of transient bargain opportunities that the bargains disappear. When I speak of rational or correct pricing, I imply no normative approval of any particular pattern but merely describe that pattern which (if it were to come into existence and were known to prevail) would continue to reproduce itself while fulfilling the postulated mean expectations in the form of α in (2) and β in (20). It would be a valuable empirical exercise to measure the α for different stocks at different times and deduce the value of β that the warrants earn *ex post* and that can rationalize the observed scatter of warrant and stock prices.

Intuition can carry us a bit further and throw light on the case where $\beta > \alpha$. With the warrant having to produce a better gain than the common, the curve for a perpetual warrant becomes strictly convex—as in Figure 1b and in contrast to Figure 1a. Furthermore, when the common price becomes very high compared to the exercise price—i.e., when X/1 is very large—the conversion value of the warrant becomes negligibly less than the common—i.e., $(X-1)/X \doteq 1$. If in the period ahead the warrant can rise at most \$1 more in price than the common rises, the warrant's gain will approach indefinitely close to the common's α. But that contradicts the assumption that $\beta > \alpha$. So for X high enough, $X > C_\infty < \infty$, it will never pay to hold the warrant in the expectation of getting $\beta > \alpha$; above this C_∞ cut-off point, the warrant must be converted. What has been demonstrated here for perpetual warrants holds *a fortiori* for finite warrants with finite T. Even sooner, at $C_T < C_\infty$, it will pay to convert since with the clock running on and running out, there will be even less advantage in holding the warrant for an additional period when the stock and it have become very large.

Linear Analysis where $\beta = \alpha \geqq 0$

If the expected yields of common and warrant are to be the same in (2) and (20), there is never any advantage in converting the warrant before the end of its life. That is

$$(22)\quad F(X,T) > F(X,0) = \mathrm{Max}(0,X\text{-}1),$$
$$T > 0;\quad \beta = \alpha \geqq 0.$$

Equation (20), postulating that the warrant have an expected gain per unit time of β, can therefore be written, for all times S,

$$(23)\quad E[Y_{t+s}(T\text{-}S) = F(X_{t+s},T\text{-}S) \,|\, Y_t(T) = F(X_t,T)]$$
$$= e^{\beta S}F(X_t,T) = \int_0^\infty F(X,T\text{-}S)\,dP(X,X_t;S)$$

or

$$(24)\quad F(x,T) \underset{s}{\equiv} e^{-\beta S}\int_0^\infty F(X,T\text{-}S)\,dP(X,x;S)$$
$$= e^{-\beta T}\int_0^\infty F(X,0)\,dP(X,x;T)$$
$$= e^{-\beta T}\int_1^\infty (X\text{-}1)\,dP(X,x;T).$$

This last integral equation provides, by a quadrature, the solution of our problem. From the fact that $P(X,x;0) = 1, X > x$ and $= 0, X < x$, it is evident that

$$\lim_{T\to 0} F(x,T) = F(x,0) = \mathrm{Max}(0,x\text{-}1).$$

We can now prove that

$$(25)\quad \lim_{T\to\infty} F(x,T) = x = F(x,\infty),\quad \beta = \alpha \geqq 0.$$

Substitute $F(x,\infty) = F(x)$ into both sides of (24) to get a self-determining integral equation for $F(x)$,

$$(26)\quad F(x) = e^{-\beta S}\int_0^\infty F(X)\,dP(X,x;S).$$

The substitution $F(x) = x$ does satisfy

(26), since by (2)

$$x = e^{-\beta S}\int_0^\infty X dP(X,x;S)$$

$$(27) \qquad = e^{-\beta S}e^{\alpha S}x = e^{(\alpha-\beta)S}x = x, \quad \beta = \alpha.$$

Any kx would also satisfy (26), but only for k = 1 do we satisfy

$$x \geqq F(x) = kx \geqq \mathrm{Max}(0,x-1).$$

To prove that the stationary solution of (26) does in fact fulfill the limit of (25), rewrite (24)

$$F(x,T) = e^{-\beta T}\int_1^\infty (X-1)\,dP(X,x;T)$$

$$(28) \qquad = e^{-\beta T}\int_0^\infty (X-1)dP(X,x;T) + e^{-\beta T}\frac{\int_0^1 (1-X)dP(X,x;T)}{\int_0^1 dP(X,x;T)}\int_0^1 dP(X,x;T)$$

$$= e^{-\beta T}e^{\alpha T}x - e^{-\beta T} + e^{-\beta T}\theta_1(x,T)\theta_2(x,T),$$

where $|\theta_1| \leqq 1$.

Obviously, if $\beta = \alpha > 0$, $F(x,\infty) = x + 0$, as was to be proved. For $\alpha = 0$

$$(29) \qquad \lim_{t\to\infty} \theta_2(x,T) = \int_0^1 dP(X,x;\infty) = 1, \text{ since } P(0+, x; \infty) \equiv 0$$

$$\lim_{t\to\infty} \theta_1(x,T) = \frac{\int_0^1 (1-x)dP(X,x;\infty)}{\int_0^1 dP(X,x;\infty)} = 1,$$

since $P(0+, x; \infty) \equiv 0$

Hence, for $\alpha=0=\beta$, $F(x,\infty)=x-1+1=x$, as required.

Now that (24) gives the explicit solution in the case $\alpha = \beta$, we can put in for $P(X,x;T)$ any specialization, such as

$$P(X,x;T) = P(X/x;T) \text{ log-normal with}$$

$$(30) \qquad P(x;T) = N(\log x;\mu t,\sigma\sqrt{t}) \text{ where}$$

$$N(y;0,1) = N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-u^2/2}\,du;$$

or

$$(31) \qquad \mathrm{Prob}\left\{\frac{X}{x} = e^{at}\right\} = e^{-bt}$$

$$\mathrm{Prob}\left\{\frac{X}{x} = 0\right\} = 1 - e^{-bt}$$

$$a - b = \alpha = \beta; \; a,b > 0.$$

For this last case, (24) calculates out to

$$(32) \qquad F(x,T) = \mathrm{Max}(0, x - e^{-aT}).$$

Note that the $\sqrt{T}$ law does not hold true here for small T, but rather, at x = 1

$$(33) \quad F(x,T) = F(1,T) = 1 - e^{-at} = 1 - 1 + aT + \text{remainder } (T^2) \doteq aT.$$

Hence, a warrant for twice the duration of a short-lived warrant should be worth about twice as much when (31) holds—even though the ratios X_{t+T}/X_t are strictly independent.

Valuation of End-of-Period Warrants

The exact solution of (24) holds only for the case $\beta = \alpha \geqq 0$. It will be shown that new formulas must handle the case of $\beta > \alpha$. However, the simple integral (24) does give a solution under all cases to the simpler case of a warrant that can be exercised *only* at the end of the period T. We might call this a "European warrant" by analogy with the "European call," which, unlike the American call that is exercisable at any time from now to T, is exercisable only at a specified terminal date.

Obviously, the additional American option of early conversion can do the owner no harm, and it may help him. Denote the rational price of a European warrant by f(x,T), in contrast to F(x,T) of the American type warrant. Then

$$(34) \qquad f(x,T) \leqq F(x,T), \; 0 \leqq T$$

and our axiom of expected gain (20) is now applicable in the form that gives the last version of (24), namely

$$(35) \qquad f(x,T) = e^{-\beta T}\int_0^\infty \mathrm{Max}(0,X-1)\,dP(X,x;T)$$

$$= e^{-\beta T}\int_1^\infty (X-1)\,dP(X,x;T), \; \beta \geqq \alpha \geqq 0.$$

Since this is the same formula as held in (24) for $F(x,T)$ when $\beta = \alpha$, we note that in such a case the American warrant's early conversion options are actually of no market value; or

$$(36) \quad f(x,T) \equiv F(x,T) \quad \text{if } \beta = \alpha.$$

When $\beta > \alpha$, (35) still holds. But now

$$(37) \qquad f(x,T) < F(x,T)$$

for all or some positive (x,T). In the log-normal case, the strong inequality must always hold.

There seems to be a misapprehension concerning this inequality. Thus some people argue that the owner of a European call or warrant can in effect exercise it early by selling the stock short, thereby putting himself in the position of the owner of an American warrant. If this view were valid there would be no penalty to be subtracted from $F(x,T)$ to get true $f(x,T)$. Such a vew is simply wrong—as wrong as the naive view that giving your broker a stop-loss order gives you the same protection as buying a put. (The fallacy here has naught to do with the realistic fact that in a bad market break your broker will not be able to execute your stop-loss order at the stipulated price; waive that point. Suppose I buy a stock at \$100 and protect it by buying (say for \$10) a six-month put on it at exercise price of \$100. You buy the stock and merely give your broker a stop-loss sell order at just below \$100. If the stock drops below \$100 at some intermediate time during the next six months, you are sold out without loss; but you do as well as I do only if the stock never subsequently rises to above \$100; and the \$10 cost of the put is precisely the market value of my opportunity to make a differential profit over you in case the stock does end up at more than \$100, after at least once dipping below \$100.) By the vector calculus that Kruizenga and I worked out for various options, after one sells a stock short and still holds a European call or warrant on it, he is not for the remainder of the time T in the position of a man who has sold out his American warrant; instead he is in the net position of holding a put on the stock. (If (1,0) and (0,1) represent holding a call and put respectively, the owner of an American warrant goes through the cycle (+1,0) and—in midstream—(−1,0), ending up with (0,0). The holder of the European warrant goes through the cycle (+1,0) and—in midstream—(−1,+1), leaving him for the remainder of the period with (0,+1).)

To see that (37) does hold when $\beta > \alpha$, recall that $F(x,T)$ cannot decrease with T. But applying to (35) the version of (24) given in (28), we can see that a long-lived European warrant does ultimately approach zero in value as $T \to \infty$. Thus, by (28) applied to $f(x,T)$,

(38)
$$f(x,T) = e^{-\beta T}\int_1^x (X-1)\,dP(X,x;T)$$
$$= e^{-\beta T}e^{\alpha T}x - e^{-\beta T} + e^{-\beta T}\theta_1\theta_2,$$
$$|\theta_1| < 1$$
$$\lim_{T\to\infty} f(x,T) = f(x,\infty) = f(x) =$$
$$e^{-(\beta-\alpha)x}x = 0, \ \beta > \alpha \geqq 0.$$

General Formula for $\beta > \alpha \geqq 0$

The last section's demonstration that $f(x,T) < F(x,T)$ when $\beta > \alpha$ provides a rigorous proof that the linear integral equation of (24) cannot apply to the proper $F(x,T)$ for this case. Hence $\beta > \alpha$ does imply that a warrant cannot possibly be worth holding at very high prices. I.e., the inequality

$$(39) \qquad F(x,T) \geqq x - 1, \quad x \geqq 1$$

must for sufficiently high x become the equality

$$(40) \quad F(x,T) = x - 1,$$
$$x > C_x(T;\beta,\alpha) < \infty, \ \beta > \alpha$$

where $\partial C_x/\partial T \geqq 0$, $\partial C_x/\partial\alpha \geqq 0$, $\partial C_x/\partial\beta \leqq 0$. (McKean's appendix also proves this fact, in 2.8 and 4.7.)

In place of the integral equation (24), we have the following basic inequality to define $F(x,T)$ where $\beta > \alpha$:

$$(41) \quad x \geqq F(x,T) \geqq \mathrm{Max}[0, x-1,$$
$$e^{-\beta S}\int_0^\infty F(X,T-S)\,dP(X,x;S)]$$

McKean's appendix terms any solution of this relation an "excessive function," and he seeks as the solution to the problem the *minimum* function that belongs to this class. Rather than arbitrarily postulate that it is the minimum function which constitutes the desired solution, I deduce from my axiom of expected gain (20) the only solution which satisfies it and which satisfies the basic inequality. It follows as a provable theorem that this does indeed give the minimum of the excessive functions. That is, any excessive function which is not the minimum will fail to earn β per unit time whenever it is being held.

How shall we find the simultaneous solution to (20) and (41)? I begin from the intuitive consideration that splitting up continuous time into small enough finite intervals will approach (from below) the correct solution for the continuous case. If a warrant can be converted only every hour, its value will be a bit less than one that can be converted at any time—less because an extra privilege is presumably worth something, only a little less because not much of a price change is to be expected in a time period so short as an hour. The approximation will be even better if we split time up into discrete minutes and still better if we use seconds. In the limit, we get the exact solution.

Let $\Delta T = h$ and define recursively in (41) for fixed h and integral n

(42)

$$F_{n+1}(x;h) = \mathrm{Max}[(0, x-1, e^{-\beta h}\int_0^\infty F_n(X;h)dP(X,x;h)]$$

$$F_0(x;h) = \mathrm{Max}(0,x-1).$$

Then

$$(43) \qquad \lim_{h=\frac{T}{n}\to 0} F_n(x;h) = F(x,T),$$

the desired exact solution to our problem as formulated by (20) and (41). In principle, by enough integrations, the degree of approximation can be made as close as we like.

The general properties of the solution can also be established by this procedure. Thus, if $P(X,x;1) = P(X/x;1)$ is a multiplicative process—or even if some weaker conditions are put on the way that P shrinks with an increase in x—we begin with a convex function $F_0(x;h)$ and end at each stage with a convex expectation function. Hence, by induction $F(x,T)$ and $F(x) = F(x,\infty)$ must be convex. ($F(x,T)$ will be strictly convex if $P(X/x;1)$ is log-normal or similarly smooth.) Where the slope $\partial F_n(x;h)\partial x$ exists it can be shown inductively that its value must lie in the closed interval [0,1], a property which must hold for $F(x,T)$. At the critical conversion point C_T, where $C_T - 1 = F(C_T,T)$, one expects the slopes of the two equal branches to be equal.

It will be instructive to work through an example in which time itself is divided into small, discrete intervals $t = 0, 1, 2, \ldots$, etc. And suppose that $P(X,x;1)$ corresponds to a simple, multiplicative random walk of martingale type, where

$$\mathrm{Prob}\left\{\frac{X_{t+1}}{X_t} = \lambda > 1\right\} = p > 0,$$

$$\mathrm{Prob}\left\{\frac{X_{t+1}}{X_t} = \lambda^{-1}\right\} = 1 - p = q > 0.$$

The gain per unit time is now given by

$$e^{\alpha} = p\lambda + q\lambda^{-1} = 1, \quad \text{where } \lambda = \frac{1-p}{p}.$$

It will help to keep some simple numbers in mind: e.g. $p = 1/3$, $q = 2/3$, $\lambda = 2$, making $\alpha = 0$ and the $[X_t]$ sequence a "fair game" or martingale, with zero net expected yield.

If β is also set equal to zero, so that it never pays to exercise the warrant, (41) reduces to the simple form (35), and we are left with the familiar partial-difference equation of the classical random walk (but in terms of $\log_\lambda X_t$, not X_t itself). Specifically, $\log_\lambda X_t/X_0$ will take on only integral values for $t > 0$; if later we make λ nearer and nearer to 1, the fineness of the grid of integral values will increase; and it will cause little loss of generality to suppose that initially $X_0 = \lambda^k$, where k is a positive or negative integer. This being assured, a two-way $F(X,m) = F(\lambda^n,m)$ can always be written as a two-way sequence F_{nm}, where m denotes non-negative integers and n integers that can be positive, negative, or zero. Corresponding to (35), we now have:

$$F_{n0} = \mathrm{Max}(0, \lambda^n - 1)$$

$$F_{n1} = pF_{n+1,0} + q F_{n-1,0}, \; p + q = 1$$

. . . .

$$F_{n,m+1} = pF_{n+1,m} + qF_{n-1,m}$$

. . . .

$$F_{n\infty} = F_n = pF_{n+1} + qF_{n-1}.$$

The last of these is an ordinary second-order difference equation with constant coefficients, whose characteristic polynomial is seen to be

$$p\sigma^2 - \sigma + (1-p) = p(\sigma - 1)(\sigma - \sigma_2),$$

$$\text{where } \sigma_2 = \frac{1-p}{p} = \lambda > 1.$$

Write the general solution for F_n as

$$F_n = c_1(1)^n + c_2\sigma_2^n$$

Since $F_n \rightarrow 0$ as $n \rightarrow -\infty$, we must have $c_1 = 0$. Since

$$\mathrm{Max}(0, X-1) = \mathrm{Max}(0, \lambda^n - 1)$$

$$\leqq F(X) = c_2\lambda^n \leqq \lambda^n = X,$$

we must have

$$c_2 = 1,\ F_n = \lambda^n,\ F(X) = X$$

verifying the general derivation of (25).

Now drop the assumption that $\alpha = 0$, but still keep $\beta = \alpha$. The above partial-difference equations are unchanged except that now (p,q) are replaced by (Bp, Bq) where

$$B^{-1} = e^{\alpha} = p\lambda + q\lambda^{-1} > 1.$$

Again it can be shown that $\sigma_2 = \lambda$ is a root of the characteristic polynomial, and that only if $(c_1, c_2) = (0, 1)$ can the boundary condition be satisfied. Again we confirm (35)'s $F(X) = X$ solution for $\alpha = \beta$.

Now let $B^{-1} = e^{\beta} > e^{\alpha} = p\lambda + q\lambda^{-1} = \Phi(\lambda) \geqq 1$.

For $m \leqq \infty$, there will exist critical integral constants n_m, equal (except for the coarseness of the integral grid) to $\log_\lambda C_m$, above which warrant conversion is mandatory. The partial-difference equations derivable from (41) now become

$$F_{n,m} = \lambda^n - 1, \quad n \geqq n_m > 0$$

$$F_{n,m} = BpF_{n+1,m-1} + BqF_{n-1,m-1}$$

$$\leqq \lambda^n - 1, n < n_m, \quad (m = 1, 2, \ldots)$$

$$F_{n,\infty} = F_n = BpF_{n+1} + BqF_{n-1}, \quad n < n_\infty$$

$$= \lambda^n - 1, \quad n \geqq n_\infty.$$

These relations define the sequence (n_m) recursively—e.g., n_1 is the lowest integer for which

$$\lambda^{n_1} - 1 \geqq Bp(\lambda^{n_1}\lambda - 1) + Bq(\lambda^{n_1}\lambda^{-1} - 1).$$

With n_1 known, we have initial conditions to the right to determine $F_{n,1}$ for $n \leqq n_1$. The difference equation for $F_{n,1}$, which can be written symbolically in terms of the operators E and E^{-1} defined by $E\,F_{n,m} = F_{n+1,m}$, $E^{-1}F_{nm} = F_{n-1,m}$ as $\Phi(E)F_{n,1} = 0$ then determines F_{n1}. With this known we determine n_2 as the smallest integer for which

$$\lambda^{n_2} - 1 \geqq B\Phi(E)F_{n,1};$$

then determine $F_{n,2}$ by $\Phi(E)F_{n,2} = 0$, etc.

The constant n_∞ can be determined along with $F_{n,\infty}$ by the following relations

$$\Phi(E)F_n = 0;\ F_n = c_1\sigma_1^n + c_2\sigma_2^n,$$

where the characteristic polynomial can be shown to be

$$\sigma\Phi(\sigma) - \sigma = Bp(\sigma - \sigma_1)(\sigma - \sigma_2),$$

where

$$0 < \sigma_1 < 1 < \lambda < \sigma_2 = \lambda^\gamma, \quad \gamma > 1.$$

If $F_n \rightarrow 0$ as $n \rightarrow -\infty$, $c_1 = 0$; to determine c_2 and $n_\infty = a$ for short, we set

$$c_2\lambda^{\gamma a}\lambda^{\gamma} = \lambda^a\lambda - 1, \quad c_2\lambda^{\gamma a} = \lambda^a - 1, \text{ or}$$

$$(\lambda^a - 1)\lambda^{\gamma} = \lambda^a\lambda - 1,\ (\lambda^{\gamma} - \lambda)\lambda^a = \gamma - 1$$

$$a = \log_\lambda(\gamma - 1) - \log_\lambda(\lambda^{\gamma} - \lambda),$$

$$c_2 = (\lambda^a - 1)\lambda^{-\gamma a},$$

where of course γ is a function of α and β through its dependence on the coefficients Bp and λ. $F_n = \lambda^{\gamma n}$ means in terms of X, the antilog of n, that $F(X) = cX^{\gamma}$, $\gamma \geqq 1$ as our first general answer.

We can always convert one-period partial difference equations into N-period equations. When we do this (p,q) are replaced by $(p^2, 2pq, q^2), \ldots$ and by $(p^N, {}_NC_1\,p^{N-1}q, \ldots, q^N)$ where ${}_NC_i$ are the familiar binomial coefficients. By the usual central limit theorem, these approach the normal distribution. But since these coefficients apply to the $F_{n,m}$, which refer to the logarithms of X, we arrive at the log-normal distribution. Hence, if we can prove that the partial-difference equation, not merely for $\Phi(E)F_n = (BpE + BqE^{-1})F_n$ but for any general set of probabilities

$$\Phi(E)F_n = B\sum_{-k}^{+r} p_j E^j F_n = F_n, \quad \sum_{-k}^{+r} p_j = 1,$$

satisfies the $F(X) = cX^{\gamma}$ power law, we have strong heuristic evidence that this will be the exact case for the log-normal case—as McKean has rigorously proved in the Appendix. The characteristic polynomial of this last becomes

$$-1 + \sigma^k \Phi(\sigma) = (\sigma - \sigma_1)(\sigma - \lambda^{\gamma})\,\Phi_2(\sigma),$$

where, as before

$$0 < \sigma_1 < \lambda \leqq \lambda^{\gamma}, \quad \gamma \geqq 1$$

and $\Phi_2(\sigma)$ is a polynomial with no roots greater than 1 in absolute value. Hence, in the general solution

$$F_n = \Sigma c_j \sigma_j^{\,n} = c_2 \lambda^{\gamma} + \text{Remainder},$$

all the c's except c_2 must vanish if $F_{-\infty} \to 0$. The value of c_2 and the critical conversion point n_c is determined just as in the simple (p, q) case. If the grid is very fine because $\lambda \to 1$, $\lambda^{n_c} = c_c = \gamma/(\gamma - 1)$ to an increasingly good approximation.

As a preview to McKean's exact result for the continuous-time case, I shall sketch the usual Bachelier-Einstein derivation of the partial differential equations of probability diffusion—of so-called Fokker-Planck type—by applying a limit process to the discrete partial difference equations. From now on consider $n = \log x$ as if it were a continuous rather than integral variable. Bachelier wrote in 1900

$$p_{n,t} = \frac{1}{2} p_{n+1,\,t-1} + \frac{1}{2} p_{n-1,\,t-1},$$

or

$$p_{n,\,t+\Delta t} = \frac{1}{2} p_{n+\Delta n,\,t} + \frac{1}{2} p_{n-\Delta n,\,t}$$

$$\frac{\Delta t}{(\Delta n)^2}\,\frac{p_{n,t+\Delta t} - p_{n,t}}{\Delta t} = \frac{1}{2}\,\frac{(p_{n+\Delta n,t} - p_{n,t})}{(\Delta n)^2} + \frac{1}{2}\,\frac{(p_{n-\Delta n,t} - p_{n,t})}{(\Delta n)^2}$$

Now if $\Delta t \to 0$, with $\Delta t/(\Delta n)^2 \to 2c^2$, we get the Fourier parabolic equation

$$c^2\,\frac{\partial p(n,t)}{\partial t} = \frac{\partial^2 p(n,t)}{\partial n^2}$$

Bachelier assumed a fair game with probabilities of unit steps in either direction equal to 1/2. If we replace (1/2, 1/2) by (p, q) so that the random walk has a biassed drift of α as its expected instantaneous rate of growth, we find $p(n - \alpha t, t)$ satisfying the above equation and hence the requisite distribution $r(n,t) \equiv p(n - \alpha t, t)$ satisfies

$$\frac{\partial^2 r(n,t)}{\partial n^2} = c^2\,\frac{\partial r(n,t)}{\partial t} + c^2\alpha\,\frac{\partial r(n,t)}{\partial n}$$

Bachelier and Einstein were talking about the diffusion of probabilities. But we have seen that the warrant prices $F_{n,t}$, now written as $F(e^n,t) = \psi(n,t)$, satisfy similar partial-difference equations, the only difference being i) that the coefficients add up to less than 1 when $\beta > \alpha$; and ii) the boundary conditions for c_t become rather complicated. Just as we had a simple second-order (partial) difference equation $E\Phi(E)F_{n,t} = EF_{n,t}$, we derive in the limit—as McKean shows in 3, and 5, drawing on the work of E. B. Dynkin—a simple (partial) second-order differential equation for $\psi(n,t)$, which in terms of $\log_e x = n$ becomes,

$$\frac{\sigma^2}{2}\,\frac{\partial^2\Psi(n,t)}{\partial n^2} + \delta\,\frac{\partial\Psi(n,t)}{\partial n} - \frac{\partial\Psi(n,t)}{\partial t} - \beta\Psi(n,t) \equiv 0, \quad \delta = \alpha - \frac{\sigma^2}{2}$$

$$\Psi(n,0) = \mathrm{Max}(0, e^n - 1)$$

$$\Psi(e^{c_t},t) = e^{c_t} - 1$$

It is understood that the equation holds for (n,t) to the left of $n = e^{c_t}$ and that $\Psi(-\infty,t) \equiv 0$. However, it is a difficult task to compute the c_t function, even using the high contact property $\partial F(c_t,t)/\partial x = 1$.

The perpetual warrant is much simpler, since then $\Psi(n,\infty) = \Psi(n)$ with $\partial\Psi(n,\infty)/\partial t = 0$, giving the *ordinary* differential equation

$$\frac{\sigma^2}{2}\Psi''(n) + \delta\Psi'(n) - \beta\Psi(n) = 0, \qquad n < c_c$$

$$\Psi(-\infty) = 0, \quad \Psi(c_c) = c_c - 1, \quad \Psi'(c_c) = e^{c_c}$$

The general solution can be written as a sum of two exponentials, in terms of the roots of the characteristic polynomial

$$\frac{\sigma^2}{2}\rho^2 + \delta\rho - \beta = \frac{\sigma^2}{2}$$

$$(\rho - \rho_1)(\rho - \rho_2), \quad \rho_1 = \gamma > 1 > \rho_2$$

If the boundary conditions are to be realized, the ρ_2 root must be suppressed and we are left with

$$\Psi(n) = (c_e - 1)\frac{e^{n\gamma}}{c_e}, \text{ or}$$

$$F(x) = (c_e - 1)\left(\frac{x}{c_e}\right)^{\gamma}$$

$$\gamma = \frac{c_e}{c_e - 1}$$

Intuitive Proofs from Arbitrage

Equation (18), which related the rational price of a warrant with any exercise price $X°$ to the formula for a warrant with $X° \gtreqless 1$, can be used directly to deduce restrictions on the way $F(x,T;X°)$ varies with $X°$. Because $F(x,T)$ has been shown to be convex with numerical slope on the closed interval $[0,1]$, (18) can deduce that the numerical slope of $F(x,T;X°)$ with respect to $X°$ must be on the closed interval $[-1,0]$—i.e.,

$$-1 \leqq \frac{F(x,T;X° + \Delta X°) - F(x,T;X°)}{\Delta X°} \leqq 0, \quad \text{or}$$

$$(44) \qquad -1 \leqq \frac{\partial F(x,T;X°)}{\partial X°} \leqq 0,$$

where the last partial derivatives, if they do not exist at certain corners, can be interpreted as either left-hand or right-hand derivatives.

One proves (44) directly by differentiating (18) with respect to $X°$, to get

$$(45) \qquad \frac{\partial F(x,T;X°)}{\partial X°} = \frac{\partial}{\partial X°}\left\{X°F\left(\frac{x}{X°},T\right)\right\} = F\left(\frac{x}{X°},T\right) - \frac{x}{X°}\frac{\partial F\left(\frac{x}{X°},T\right)}{\partial(x/X°)}$$

That the right-hand expression in (45) is non-positive follows directly from the definition of convexity of $F(x,T)$ when $F(0,T) \equiv 0$. That it is not algebraically less than -1 follows from the fact that $F(x,T) \geqq \text{Max}(0,x-1)$.

Intuitive economic arguments provide an alternative demonstration that

$$(46) \qquad -1 \leqq \frac{\partial F(x,T;X°)}{\partial X°} \leqq 0$$

An increase in the exercise price $X°$ must, if anything, lower the value of the warrant since it then entails a higher future payment. But a fall of \$1 in $X°$ can never be worth more than \$1, since stapling a \$1 bill to a warrant with $X°$ exercise price is a possible way of making it the full equivalent of a warrant exercisable at $X° - \$1$. Hence, we have established (46).

The condition for high contact at a conversion point C_T, namely $\partial F(x,T)/\partial x \to 1$ as $x \to C_T$, seems intuitively related to realization of left-hand equality in (46) as $x \to C_T/X°$, which in turn seems intuitively related to the probability that, when x is already near C_T, x will be reaching C_t in a sufficiently short future time. For the log-normal Brownian motion of (30) and the special case of (31), these conditions for high contact will be realized. But for any solution of the Chapman-Kolmogorov equation (4) of log-Poisson type, like that discussed by McKean and involving jumps, high contact will definitely fail. If we rule out combinations of Poisson jumps, only (30) and (31) and combinations of them like that shown in (16) would seem to be relevant. For them high contact is indeed ensured. And for both of these types an exact power-of-x solution for the perpetual warrant has been shown by McKean to hold.

Final Exact Formula for Perpetual Warrant in Log-Normal Case

McKean has proved in (3.0) the following exact smooth formula for $F(x,\infty) = F(x)$, for the log-normal case

$$(47) \qquad F(x) = \frac{(\gamma-1)^{\gamma-1}}{\gamma^{\gamma}}x^{\gamma} = (c-1)\left(\frac{x}{c}\right)^{\gamma}, \quad x \leqq c = \frac{\gamma}{\gamma-1} > 1$$

x	0	.25	.50	.75	1.0	1.25	1.50	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$\frac{1}{4}x^2$	0	.016	.062	.141	.250	.391	.562	.766	1.0*	1.5*	2.0*	2.5*	3.0*	3.5*	4.0*
$2\left(\frac{x}{3}\right)^{3/2}$	0	.048	.136	.250	.385	.538	.707	.891	1.088	1.521	2.0*	2.5*	3.0*	3.5*	4.0*
$3\left(\frac{x}{4}\right)^{4/3}$	0	.074	.188	.322	.472	.636	.811	.996	1.190	1.603	2.044	2.511	3.0*	3.5*	4.0*
$4\left(\frac{x}{5}\right)^{5/4}$	0	.094	.225	.373	.535	.707	.888	1.078	1.572	1.682	2.112	2.561	3.026	3.506	4.0*

Figure 3 Rational Price for Perpetual Warrant in Log-Normal Model

[Explanation: $x = X/X°$, the common stock price ÷ exercise price. $y = Y/Y°$, warrant price ÷ exercise price is given by $y = (c-1)(x/c)^\gamma$ where $\gamma = c/(c-1)$; value of γ depends on α/σ^2 and β/σ^2 as given in Equation (48).]

* Warrant at conversion value, $x - 1$.

$$= x - 1,\ x \geqq c,\ \gamma = \frac{c}{c-1} > 1.$$

This has the nice property of high contact, with $F'(c) = 1$ from either direction. Examples of (47) for different values of γ would be

$$F(x) = 3\left(\frac{x}{4}\right)^{4/3}, \quad \gamma = 4/3,\ c = 4$$

$$F(x) = 2\left(\frac{x}{3}\right)^{3/2}, \quad \gamma = 3/2,\ c = 3$$

$$F(x) = \frac{1}{4}x^2, \quad \gamma = 2,\ c = 2$$

The last of these formulas has been proposed, in different notation, on a purely *ad hoc* empirical basis by Guigère.[11]

I append a brief *table of values* (Figure 3) of $F(x)$ for what would seem to be empirically relevant values of γ.* Figure 4 plots as straight lines on double-log paper $F(x)$ for various values of γ.

To relate γ to α, β, and the dispersion parameter σ^2 in the log-normal distribution, I rewrite McKean's formula for γ (in the Appendix) as

$$(48)\quad \gamma = \left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right) + \sqrt{\left[\frac{1}{2} + \frac{\alpha}{\sigma^2}\right]^2 + 2\left[\frac{\beta}{\sigma^2} - \frac{\alpha}{\sigma^2}\right]}.$$

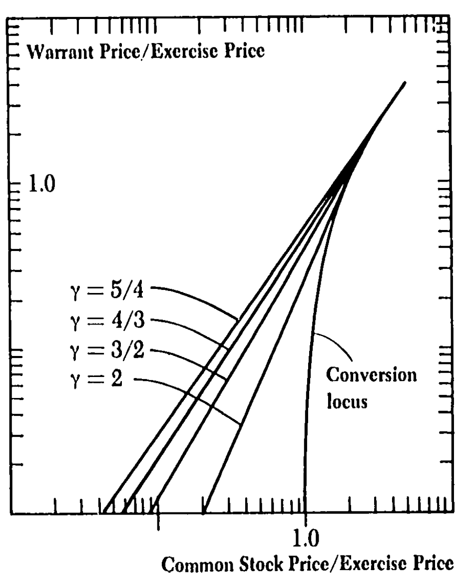

Figure 4 Rational Price for Perpetual Warrant in Log-Normal Model

[11] Ref. [17]. The notation there, of course, needs to be related to my notation involving $\frac{X}{X°}$ and $\frac{Y}{Y°}$, as in Figure 3.

*Acknowledgment is made to F. Skidmore for these computations.

That γ is a function of $(\alpha/\sigma^2, \beta/\sigma^2)$ follows from the invariance of the problem under transformations of the unit used to measure time. Similar ratios of parameters occur in the log-Poisson process and the multiplicative-translation-with-absorption process of (15).

It is instructive to hold (α,β) fixed in (48), and examine how γ varies with the dispersion parameter σ^2 of the log-normal process for the stock. When $\sigma^2 \to \infty$, the difference $(\beta - \alpha)/\sigma^2 \to 0$ and $\gamma \to 1$, the case where the warrant never gets prematurely converted. Such a large value for the dispersion parameter σ^2 would create a very large α if the drift of log X_t were not strongly negative. Any such negative drift implies that it is "almost certain" that the holder of the stock will be "eventually" ("almost completely") ruined—even though the stock does have a positive mean capital gain. Note the tricky statement involving a triple limit, as in the earlier theorem on (virtually) certain (relative) ruin.

We'll see in (50) that $\gamma = \sqrt{\beta/\alpha}$ when $\sigma^2 = 2\alpha$ and there is no drift at all to log X_t and hence to X_t. In this knife-edge case of Osborne, where the geometric mean of future X_{t+T} just equals X_t, the probability of a future capital loss (or gain) is exactly one half. At the other limit, where the dispersion $\sigma^2 \to 0$, we put $(\alpha/\sigma^2, \beta/\sigma^2) = (\infty,\infty)$ in (48) and find $\gamma \to \beta/\alpha$. This can be verified by substituting into $y = (c - 1)(X/c)^\gamma$ the now-*certain* path $X(t) = X_0 e^{\alpha T}$ and deducing $Y(t) = Y_0 e^{\beta t} = Y_0 e^{\gamma\alpha T}$, with $\gamma = \beta/\alpha$.

To estimate γ empirically, one might regress log warrant price against log common price, γ being the regression coefficient. Then α might be estimated statically by calculating the mean percentage gain per unit time of the common, or by computing $E[X_{t+1}]/X_t = e^\alpha$. Then β will be determined by the formula (48) for γ once one has an estimate of σ^2. Since σ is the standard deviation of $\log(X_{t+1}/X_t)$, it can be estimated from the sample variance of this last variate. The consistency of the model with the facts could then be checked by calculating β separately as the mean value of the warrant's gain, or by

$$E[Y_{t+T}]/Y_T = e^{\beta T}$$

where T is always less than the time after t when it pays to convert the warrant. A further check on the log-normality model comes from the fact that, when the "instantaneous variance per unit of time" of X_t is σ^2, the instantaneous variance for unit time of Y_t should work out to be $\gamma^2\sigma^2$, greater than σ^2 by the factor $\gamma^2 > 1$.

I am not presenting any empirical results here. But I shall draw upon some findings of others by way of illustrating the theory. (Incidentally, they suggest remarkably high β/α, giving the warrants a suspiciously favorable return.)

Osborne[12] finds some empirical warrant for his theoretically dubious axiom that log X_t takes an unbiased random walk, with neither upward nor downward drift. If μt represents the net drift of log X_t, we have

$$(49) \qquad \mu = \alpha - \frac{\sigma^2}{2} = 0, \quad \frac{\alpha}{\sigma^2} = \frac{1}{2}.$$

Substituting these values into (48) gives

$$(50) \qquad \gamma = \sqrt{\frac{\beta}{\alpha}}, \quad \text{when } \frac{\alpha}{\sigma^2} = \frac{1}{2}.$$

Osborne and many investigators report average capital gains on a stock of three to five per cent per year. So set $\alpha = .04$. Finally Giguère in the cited paper[13] infers $\gamma = 2$ from empirical scatters of perpetual warrant prices against their common stock prices. (My casual econometric measurements suggest $\gamma = 2$ is much too high: these days one can rarely buy a long-lived warrant for only one-fourth of the common when the common is selling near its exercise price. But accept $\gamma = 2$ for the sake of the demonstration.) Combining $\mu = 0$, $\alpha = .04$, $\gamma = 2$, we get for the mean return per year for holding the warrant no less than 16 per cent!—i.e., $\beta = \gamma^2 \alpha = 4(.04) = .16$.

This does seem to be a handsome return, and one would expect it to be whittled away over time—unless people are exceptionally averse to extra risk. The high β return would be whittled away as people bid up the prices of perpetual warrants until they approached the value of the common stock itself—at which point $\beta =$

[12]Ref. [3], p. 108.

[13]Ref. [17].

α, $\gamma = 1$, and $c = \infty$. There is no other way. Yet this does not seem to happen. Why not? One obvious explanation is that whenever a stock pays a regular dividend of δ per period, β will, taxes aside, naturally come to exceed α by at least that much. But there are stocks that pay no dividend which still sell much above their perpetual warrants. Perhaps a departure from our assumption of a stationary time series, in the form of a supposition that there will later be a regular dividend, can help explain away the paradox. Coming events do cast their shadow before them.

I should like now to sketch a theory to explain why $\beta - \alpha$ cannot become too large. If $\beta > \alpha$ so that $\gamma > 1$, hedging will stand to yield a sure-thing positive net capital gain (commission and interest charges on capital aside!). This follows from the concept of leverage as curvature in Figure 2. Let the stock be initially at X_0 with the warrant at $F(X_0) = Y_0$. Then buying \$1 long of the warrant and selling \$1 short of the common gives the new hedged variate $Z = Y/Y_0 - X/X_0$. Whether X goes up or down, Z is sure to end up greater than 1, with a positive gain. Indeed, its expected gain per unit time is $\beta - \alpha$. But there will be a variance per unit time around this mean value that works out to $(\gamma - 1)^2\sigma^2$. This variance will be quite small when γ is near to 1, but with $\gamma \doteq 1$ it is likely that the difference $\beta - \alpha$ will also be small.

In the example worked out earlier from the data of Osborne and Giguère, a hedger would have the same variance as would a buyer of the common stock; but instead of earning 4 per cent a year, he would earn 12 per cent a year. And, commissions aside, he would have no risk of a positive loss. This would seem like almost too much of a good thing. Under the stock exchange rules, I believe he would have to put up about the same amount of money as margin to engage in the hedged transaction as to buy a dollar's worth of the warrant or stock outright; he would not need margin money for each side of the hedged transaction. So he would have to reckon in the opportunity cost of the safe interest rate per unit time of money itself, ρ. Presumably though, the buyer of the common stock has already felt that its $\alpha = .04$ return was adjusted to compensate for that ρ. (If the stock pays a percentage dividend, δ, the excess $\beta - \alpha$ includes compensation for δ, ρ and for extra riskiness. Actually, if the excess of β over α comes only from the fact of the dividend δ, there is no advantage to be gained from the hedge; this is because the man who sells the common short must make good the dividend, and that will reduce the apparent profit of the hedge to zero. Hence in what follows, I deal only with the excess of β over α that is unrelated to dividends, and I ignore all dividends.)

If hedging arbitrage *alone* is counted on to keep $\beta - \alpha$ small, under present margin requirements we should expect $\beta - \alpha = \rho$ if riskiness were not a consideration. Since there is some aversion to dispersion around the mean gain from the hedge, we should not expect from hedging arbitrage *alone* that $\beta - \alpha < \rho$. On the other hand, if people are risk averters and $\gamma < 2$, as seems realistic, it is hard to see how one could get $\beta - \alpha > \alpha$, since people would shift from holding X outright to holding a hedged position Z if the latter had the greater return, less variance, and no chance of loss. One could, in principle, learn from stock exchange records how much hedging is in fact being done, since a rational hedger will minimize margins by dealing with one firm on both sides of his hedge. It is my impression that not much warrant hedging is in fact done, although in convertible bonds there does seem to be a greater volume of hedging. Still if γ and $\beta - \alpha$ threatened to become too large, potential hedgers would become actual hedgers. Hence, the limits derived above do have some relevance, particularly

$$\beta - \alpha < \alpha. \tag{51}$$

Conclusion

The methods outlined here can be extended by the reader to cases of calls and puts, where the dividend receives special treatment different from the case of warrants, and to the case of convertible bonds.

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Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem of Mathematical Economics*

Henry P. McKean, Jr., M.I.T.

1 Introduction

Paul A. Samuelson has developed in the preceding paper a model of warrant pricing from the economic standpoint; the purpose of the present article is to add some mathematical complements.

Samuelson supposes that the motion of the price x of the common stock is a (multiplicative) differential process; this means that for each $s \geqq 0$, the (scaled) future motion $x(t+s)/x(s) : t \geqq 0$ is independent of the past $x(t) : t \leqq s$ and has the same statistics as $x(t)/x(0) : t \geqq 0$. Define $P_1(B)$ $[E_1(f)]$ to be the chance of the event B [expectation of the function f] for prices starting at $x(0) = 1$ and impose the

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condition $E_1(x) < \infty$. $E_1(x) = e^{\alpha t}$ follows and it is assumed that $\alpha \geq 0$.

Define $h = h(t, \xi)$° to be the "correct" price of a warrant to purchase the common stock at unit price, as a function of the time of purchase $t \geq 0$ and of the current price $\xi \geq 0$, subject to the additional condition that the warrant appreciate at the rate $\beta \geq \alpha$ up to such time as it becomes unprofitable still to hold it. The problem of computing h has the following mathematical expression: *find the smallest solution* $f = h$ *of*

$$f(t, \xi) \geq e^{-\beta s}E_1[f(t-s, \xi x(s))] \qquad (s \leq t, \xi \geq 0)$$

that lies above $(\xi - 1)^+ \equiv$ *the greater of* $\xi - 1$ *and* 0; the simpler problem of finding the "correct" price $h(\infty, .)$ of the perpetual warrant can be expressed in the same language as follows: *find the smallest solution* $f = h$ *of*

$$f(\xi) \geq e^{-\beta t}E_1[f(\xi x(t))] \qquad (t \geq 0, \xi \geq 0)$$

that lies above $(\xi - 1)^+$.

The existence of h is proved and its simplest properties discussed in sections 2 and 4 below: if $\beta > \alpha$, h turns out to be an increasing convex function of ξ up to a point $\xi = c(t) > 1$ [the corner], to the right of which it coincides with $\xi - 1$; c and h increase with time to $c(\infty) < \infty$ and $h(\infty, \xi) < \xi$. $h(\infty, .)$ is computed in section 3 for a (multiplicative) Brownian motion of prices [$h = (c-1)(\xi/c)^\gamma$, $c = \gamma/(\gamma - 1)$], and also for a (multiplicative) Poisson process of prices [h = a broken line]; and in section 5, h is computed for $t \leq \infty$ and a (multiplicative) translation of prices with possible absorbtion at 0. A partial solution of the problem for $t < \infty$ and a (multiplicative) Brownian motion of prices is described in section 6: it leads to a free boundary problem for the heat equation, the free boundary being a solution of an unfortunately intractible integral equation due to I. I. Kolodner [4].

An unsolved problem is to find a nice condition on the prices that will make $h^-(c) = 1$. $h^-(c)$ is the left slope at the corner. $h^-(c) \leq 1$ is automatic. Samuelson has conjectured that this will be the case if $Q = P_1[x(t) \leq 1, t \downarrow 0] = 0$ [the alternative is $Q = 1$], but I could not prove it. Another inviting unsolved problem is to discuss the integral equation for the free boundary of section 6.

I must not end without thanking Professor Samuelson for posing me this problem and for several helpful conversations about it.

2 Perpetual Warrants

Consider a (multiplicative) differential process with sample paths $t \to x(t) = x(t+) \geq 0$, probabilities $P_1(B)$, and expectations $E_1(f)$, starting at $x(0) = 1$, *i.e.*, let $P_1[x(0) = 1] = 1$ and, conditional on $x(t_1) > 0$, let $x(t_2)/x(t_1)$ be independent of $x(s) : s \leq t_1$ and identical in law to $x(t_2 - t_1)$ for each choice of $t_2 \geq t_1 \geq 0$. $P_a(B)$ and $E_a(f)$ denote probabilities and expectations for the motion starting at $x(0) = a \geq 0$; this motion is identical in law to $[ax, P_1]$; *esp.*, $P_0[x(t) = 0, t \geq 0] = 1$ and $P_a[x(t) < b] = P_1[x(t) < b/a]$ for $a, b > 0$. $[x, P_.]$ *begins afresh at stopping times.* A *stopping time* is a non-negative function $\mathit{t} \leq \infty$ of the sample path, such as $\mathit{t} = 1$ or the exit time $\mathit{t} = \inf(t : x > 1)$, such that for each $t \geq 0$ the event $(\mathit{t} < t)$ depends upon $x(s) : s \leq t$ alone. *Beginning afresh* means that if $B_{\mathit{t}+}$ is the field of events B such that $B \cap (\mathit{t} < t)$ is measurable over $x(s) : s \leq t$, then *conditional on the present* $a = x(\mathit{t})$ *and on the event* $\mathit{t} < \infty$, *the future* $x(\mathit{t} + t) : t \geq 0$ *is independent of the past* $B_{\mathit{t}+}$, *and identical in law to* $[x, P_a]$:

$$P_.[x(\mathit{t} + t)\epsilon db | B_{\mathit{t}+}] = P_a[x(t)\epsilon db] \quad \textit{if } \mathit{t} < \infty;$$

see G. Hunt[1] for a complete explanation of stopping times for (additive) differential processes. WARNING: The reader is cautioned that the italic letter *t* [stopping time] must be carefully distinguished from the roman letter t [a constant time]; italic *x* [stopping point $x(\mathit{t})$] must likewise be distinguished from roman x [sample path].

$E_1(x) = f(t)$ is a solution of $f(t-s)f(s) = f(t)$ $(s \leq t)$ and $0 < f \leq \infty$, so $E_1(x) = e^{\alpha t}$ for some $-\infty < \alpha \leq \infty$. $P_1[x = 1, t \geq 0] = 0$ or 1 because $P_1[x(s) = 1, s \leq t] = f(t)$ is a solution of $f(t-s)f(s) = f(t)$ $(s \leq t)$ and $0 \leq f \leq 1$, so that $f = e^{-\gamma t}$ for some $0 \leq \gamma \leq \infty$ and $f(\infty) = 0$ or 1 according as $\gamma > 0$ or not. $P_1[x = 1, t \geq 0] = 0$ *is assumed below.*

A non-negative function $f \not\equiv \infty$ defined on $[0, \infty)$ is $(\beta-)$ excessive if $e^{-\beta t}E_.[f(x)] \uparrow f$ as $t \downarrow 0$; in this language *the problem of the perpetual warrant is to find the smallest excessive function* $h \geq (\xi - 1)^+$ *in case* $\infty > \beta \geq \alpha \geq 0$. h is constructed and its simplest properties derived in a series of brief articles.

1. *Define* $h^0 = (\xi - 1)^+$ *and* $h^n = \sup_{t \geq 0} e^{-\beta t} E_.[h^{n-1}(x)]$ *for* $n \geq 1$; *then* $(\xi - 1)^+ \leq h^n \uparrow h \leq \xi$ *as* $n \uparrow \infty$.

PROOF: $h^{n-1}(\xi) \leq h^n(\xi) = \sup_{t \geq 0} e^{-\beta t}E_\xi[h^{n-1}(x)]$

$$\leq \sup_{t \geq 0} e^{-\beta t}E_\xi(x) = \sup_{t \geq 0} e^{-\beta t}e^{\alpha t}\xi = \xi$$

if $h^{n-1} \leq \xi$, and the obvious induction completes the proof.

°Samuelson's notation for this is F(X,T).

[1]See ref. [3].

2. h *is increasing, convex (and so continuous), and its slope is* $\leqq 1$.

PROOF: $h^n(\xi) = \sup_{t \geqq 0} e^{-\beta t} E_1[h^{n-1}(\xi x)]$ inherits all the desired properties from h^{n-1}; now use induction and let $n \uparrow \infty$.

3. h *is the smallest excessive function* $\geqq (\xi - 1)^+$.

PROOF: $e^{-\beta t} E.[h(x)] \leqq h$ is obvious from 1, and since $h \epsilon C[0, \infty)$ (2), an application of Fatou's lemma implies $\lim_{t \downarrow 0} e^{-\beta t} E.[h(x)] \geqq E.[\liminf h(x)] = h$, completing the proof that h is excessive. Also, $h \geqq (\xi - 1)^+$, and if j is another such excessive function, then the obvious induction supplies us with the underestimate $j \geqq h^n \uparrow h (n \uparrow \infty)$.

4. $h = (\xi - 1)^+$ *to the right of some point* $1 < c \leqq \infty$. $h > (\xi - 1)^+$ *to the left.*

PROOF: Given $s \leqq t$ and $a, b > 0$,

$$P_1[x(t) \geqq ab] \geqq P_1[x(s) \geqq a, x(t)/x(s) \geqq b] = P_1[x(s) \geqq a] P_1[x(t-s) \geqq b]$$

so that $P_1[x(nt) \geqq d^n] \geqq P_1[x(t) \geqq d]^n$, and *either* $P_1[x \leqq 1] \equiv 1$ $(t \geqq 0)$ violating $P_1[x = 1, t \geqq 0] = 0$ (use $E_1(x) \geqq 1$) *or* $P_1[x(nt) \geqq d^n] > 0$ for some $t > 0$, $d > 1$, and each $n \geqq 1$. But in the second case, $h \geqq e^{-\beta nt} E.[(x(nt) - 1)^+] > 0$ for $n \uparrow \infty$, and the statement follows from 2.

5. $h \equiv \xi$ *if* $\beta = \alpha \geqq 0$.

PROOF: $\xi \geqq h \geqq e^{-\beta t} E_\xi[x - 1] = \xi(1 - e^{-\beta t}) \uparrow \xi$ as $t \uparrow \infty$ if $\beta = \alpha > 0$, while if $\beta = \alpha = 0$, then $E_\xi[(x-1)^+] \geqq (\xi - 1)^+$ so that $h = \lim_{t \uparrow \infty} E.[(x-1)^+] = E.[h(x)]$. Because h is convex (2), its 1-sided slope h^+ is an increasing function,

$$h(\xi) = h(1) + (\xi - 1) h^+(1) + \int_1^\xi [h^+(\eta) - h^+(1)] d\eta,$$

and putting $\xi = x$ and taking expectations (E_1) on both sides, it follows that $h^+ \equiv h^+(1)$ between $0 < a < 1$ and $b > 1$ if $0 < P_1[x \leqq a] P_1[x \geqq b]$ for some $t > 0$. But $P_1[x(nt) \geqq d^n] > 0$ for some $t > 0$, $d > 1$, and each $n \geqq 1$ as in 4, and using the same method, it is also possible to make $P_1[x(nt) \leqq d^{-n}] > 0$ for the same $t > 0$, some (perhaps smaller) $d > 1$, and each $n \geqq 1$ (use $E_1(x) = 1$). $h^+ \equiv h^+(1)$ is immediate and $h \equiv \xi$ follows from the bounds $\xi - 1 \leqq h \leqq \xi$ and the fact that $h(0+) = 0$.

WARNING: $\beta > \alpha \geqq 0$ *until the end of the next section.*

6. *Given a closed interval* $0 < a \leqq \xi \leqq b < \infty$ *with exit time* $\mathfrak{t} = t_{ab} \equiv \inf(t : x < a$ *or* $x > b)$ *and exit place* $\mathfrak{x} = x(\mathfrak{t})$, $P.[\mathfrak{t} < \infty] \equiv 1$ *and* $j = j_{ab} \equiv E.[e^{-\beta \mathfrak{t}} h(\mathfrak{x})]$ *lies under* h.

PROOF adapted from E. B. Dynkin[2]: $P.[\mathfrak{t} < \infty] \equiv 1$ since in the opposite case, $0 < p(\xi) = P_\xi[a \leqq x \leqq b, t \geqq 0]$ for some $a \leqq \xi \leqq b$, and putting $p_{ab} = \sup_{ab} p(\xi)$, the bound $p(\xi) = E_\xi[p[x(n)], a \leqq x \leqq b, t \leqq n] \leqq p_{ab} P_\xi[a \leqq x \leqq b, t \leqq n]$ decreases to $p_{ab} p(\xi)$ as $n \uparrow \infty$, proving $p_{ab} = 1$. But $p_{ab} = \sup_{ab} P_1[a/\xi \leqq x \leqq b/\xi, t \geqq 0] \leqq P_1[a/b \leqq x \leqq b/a, t \geqq 0]$, and this cannot be 1 without violating the estimate $P_1[x(nt) \geqq d^n] > 0$ of 4. Define $G_\gamma f = E.\left[\int_0^{\mathfrak{t}} e^{-\gamma t} f(x) dt\right]$ for non-negative f and $\gamma \geqq 0$. $G_\gamma h \equiv u < \infty$ if $\gamma \geqq \beta$ and $G_\gamma = G_\beta[1 + (\beta - \gamma) G_\gamma]$, so that if $\gamma \geqq \beta$ and $v = h + (\beta - \gamma) u$, then $u = G_\beta v = E.\left[\int_0^{\mathfrak{t}} e^{-\beta t} v(x) dt\right]$.

Because of $v \geqq h + (\beta - \gamma) \int_0^{\mathfrak{t}} e^{-\gamma t} dt \, e^{\beta t} h = 0$, it follows that

$$u \geqq E.\left[\int_t^{\mathfrak{t}} e^{-\beta t} v(x) dt\right] = E.\left[e^{-\beta t} \int_0^{\mathfrak{t}} e^{-\beta t} v[x(t + t)] dt\right],$$

and since x begins afresh at the stopping time $\mathfrak{t}$ while $\mathfrak{t}$ itself is measurable over $\mathbf{B}_{\mathfrak{t}+}$,

$$u \geqq E.[e^{-\beta \mathfrak{t}} G_\beta v(\mathfrak{x})] = E.[e^{-\beta \mathfrak{t}} u(\mathfrak{x})]$$

with $\mathfrak{x} = x(\mathfrak{t})$. Now use the fact that $(\gamma - \beta) u \uparrow h$ as $\gamma \uparrow \infty$.

7. j *is excessive.*

PROOF: Because x begins afresh at time $t \geqq 0$,

$$e^{-\beta t} E.[j(x)] = E.[e^{-\beta \mathfrak{t}^\circ} h(\mathfrak{x}^\circ)] \equiv j^\circ$$

with $\mathfrak{t}^\circ$ defined as the next exit time from $a \leqq \xi \leqq b$ after time t and $\mathfrak{x}^\circ = x(\mathfrak{t}^\circ)$. Using the notation and method of proof of 6,

$$E.[e^{-\beta \mathfrak{t}^\circ} u(\mathfrak{x}^\circ)] = E.\left[\int_{\mathfrak{t}^\circ}^{\infty} e^{-\beta t} v(x) dt\right] \leqq E.\left[\int_{\mathfrak{t}}^{\infty} e^{-\beta t} v(x) dt\right] = E.[e^{-\beta \mathfrak{t}} u(\mathfrak{x})],$$

and since $(\gamma - \beta) u \uparrow h$ as $\gamma \uparrow \infty$, it follows that $j^\circ \leqq j$. But also $\mathfrak{t}^\circ \downarrow \mathfrak{t}$ as $t \downarrow 0$ and $x(\mathfrak{t}+) = x(\mathfrak{t})$, so Fatou's lemma implies

[2] See ref. [2].

$$\lim_{t \downarrow 0} j^\circ \geq E.[\liminf e^{-\beta t^\circ} h(x^\circ)] = j,$$

completing the proof.

8. $c < \infty$ *and* $E.[e^{-\beta t}h(x)] \equiv h$ *with* $t = \min(t : x \geq c)$, $x = x(t)$, *and* $e^{-\beta t}h \equiv 0$ *if* $t = \infty$, *in case* $P_1[t_{ab} \downarrow 0$ *as* $a \uparrow 1$ *and* $b \downarrow 1] = 1$.

PROOF: Define for the moment $t = t_{ab}$ and $x = x(t_{ab})$. Because x is differential and $h > (\xi - 1)^+$ near $\xi = 1$, it is possible to choose $a < 1 < b$ so as to make $\min_{ab} E_\xi[e^{-t-|x-1|}]$ so large that $j_{ab} \geq (\xi - 1)^+$. But $j_{ab} \leq h$ is excessive while h is the smallest excessive function $\geq (\xi - 1)^+$, so $j_{ab} \equiv h$ for this choice of $a < 1 < b$. Given 2 overlapping closed intervals $a_1 \leq \xi \leq b_1$ and $a_2 \leq \xi \leq b_2$ with $0 < a = a_1 < a_2 < b_1 < b_2 \equiv b < \infty$ and corresponding functions $j \equiv h$, it is to be proved that $j_{ab} \equiv h$ also. Consider for the proof paths starting at $a_1 \leq x(0) = \xi \leq b_1$ and define stopping times

$t_1 =$ *the exit time from* a_1b_1,

$t_2 = t_1$ *or the next exit time from* a_2b_2 *according as* $t_1 = t_{ab}$ *or not,*
$t_3 = t_2$ *or the next exit time from* a_1b_1 *according as* $t_2 = t_{ab}$ *or not,*
etc.
$t_1 \leq t_2 \leq$ *etc.* $\leq t_n$ is constant $(= t_\infty = t_{ab})$ from some smallest $n = m$ on, and putting $x_n = x(t_n)$ $(n \leq \infty)$, a simple induction justifies

$$h(\xi) = E_\xi[e^{-\beta t_n}h(x_n)] = E_\xi[e^{-\beta t_\infty}h(x_\infty), n \geq m] + E_\xi[e^{-\beta t_n}h(x_n), n < m].$$

As $n \uparrow \infty$, this tends to j_{ab} since $P_\xi[m < \infty] = 1$ while $h(x_n) \leq b < \infty$ on $(n < m)$. $j_{ab} \equiv h$ follows at once. Now choose $0 < a < 1 < b < c$ so that $j_{ab} \equiv h$. Repeating the first part of the proof, it is clear that the function j associated with a small neighborhood of b is identical to h, and using the second part, it follows that $j_{ab} \equiv h$ for a little bigger b. Because a can be diminished for the same reason, it is clear that if $0 < b < c$ (or if $b = c$ in case $c < \infty$), then it is possible to find closed intervals $0 < a_n \leq \xi \leq b_n < b$ increasing to $0 < \xi < b$ with $j_n \equiv h$. But for paths starting at $0 \leq x(0) = \xi \leq b$ and $n \uparrow \infty$, the exit times t_n from $a_n \leq \xi \leq b_n$ increase to the exit time $t_\infty = \min(t : x = 0$ *or* $x \geq b)$ while $x_n = x(t_n)$ tends to $x_\infty = x(t_\infty)$,° so

$$h(\xi) = \lim_{n \uparrow \infty} j_n(\xi) = \lim_{n \uparrow \infty} E_\xi[e^{-\beta t_n}h(x_n)] = E_\xi[e^{-\beta t_\infty}h(x_\infty)]$$

because of the bound

$$e^{-\beta t_n}h(x_n) = e^{-\beta t_\infty}h(x_\infty) \quad \text{if } x_n \geq b$$
$$< b \quad \text{if } x_n < b,$$

and to complete the proof, it suffices to replace t_∞ by $t = \min(t : x \geq b)$ and to prove $c < \infty$. As to t_∞, since $h(0) = 0$, $h = E.[e^{-\beta t_\infty}h(x_\infty), x_\infty \geq b] = E.[e^{-\beta t}h(x)]$ with $x = x(t)$ and the convention $e^{-\beta t}h \equiv 0$ if $t = \infty$. As to the proof that $c < \infty$, if $c = \infty$, then $h = E.[e^{-\beta t_n}h(x_n)]$ with $t_n = \inf(t : x > n)$ and $x_n = x(t_n)$. Because

$$\xi - 1 \leq h(\xi) \leq E_\xi[e^{-\beta t_n}x(t_n)] = E_1[e^{-\beta t_{n/\xi}}\xi x(t_{n/\xi})]$$

for $n > \xi$, $1 \leq E_1[e^{-\beta t_2}x(t_2)]$ as follows on putting $n/\xi = 2$ and letting $n \uparrow \infty$. Because $\beta > \alpha$, $E_1[e^{-\beta t_2}x(t_2)] < E_1[e^{-\alpha t_2}x(t_2)]$, and adapting the proof of 6 to the $(\alpha-)$ excessive function $f \equiv \xi$, one finds $E_\xi[e^{-\alpha t_2}x(t_2)] \leq \xi$. But this leads to a contradiction: $1 < E_1[e^{-\alpha t_2} x(t_2)] \leq 1$.

9. $c < \infty$ *and* $E.[e^{-\beta t}h(x)] \equiv h$ *with* $t = \min(t : x \geq c)$ *and* $x = x(t)$ *in general.*

PROOF: 8 covers the case $P_1[t_{ab} \downarrow 0$ *as* $a \uparrow 1$ *and* $b \downarrow 1] = 1$; otherwise, $P_1\left(\lim_{\substack{a \uparrow 1 \\ b \downarrow 1}} t_{ab} > 0\right) = 1$ according to Kolmogorov's 01 law, so the particle moves by *jumps* with exponential holding times between. Consider the modified motion $x^\circ = e^{\epsilon t}x$ with so small a positive ϵ that $\beta > \alpha^\circ = \alpha + \epsilon$ and let h° and c° be the analogues of h and c. Because $e^{-\beta t}E.[h^\circ(x)] \leq e^{-\beta t}E.[h^\circ(x^\circ)] \leq h^\circ$, it is clear that $h^\circ \geq h$ and $c^\circ \geq c$. As $\epsilon \downarrow 0$, $h^\circ \downarrow j \geq h$ and $c^\circ \downarrow b \geq c$. Because x° satisfies the conditions of 8, $c^\circ < \infty$ (*esp.*, $c \leq b < \infty$) and $h^\circ = E.[e^{-\beta t^\circ}h^\circ(x^\circ)]$ with $t^\circ = \min(t : x^\circ \geq b)$ and $x^\circ = x^\circ(t^\circ)$. Now an unmodified path starting at $x(0) = \xi < b$ *jumps* out of $[0, b)$ landing at $x \geq b$; this means that $t^\circ = t$ and $x^\circ = e^{\epsilon t}x$ for $\epsilon \leq 1/n$ for some n depending upon the path, so $e^{-\beta t^\circ}h^\circ(x^\circ) \downarrow e^{-\beta t}j(x)$ as $\epsilon \downarrow 0$ for a class of paths with as large a probability as desired, while on the complement of this class $e^{-\beta t^\circ}h^\circ(x^\circ) \leq e^{-\beta t^\circ}e^{\epsilon t^\circ}b \leq b$. Because $h = j = \xi - 1$ $(\xi \geq b)$ and $x \geq b$, it follows that

$$j(\xi) = \lim_{\epsilon \downarrow 0} E_\xi[e^{-\beta t^\circ}h^\circ(x^\circ)] = E_\xi[e^{-\beta t}j(x)] = E_\xi[e^{-\beta t}h(x)] \leq h$$

for $\xi < b$, *i.e.*, $j \equiv h$, and since $b \geq c$, the result follows after a moment's reflection.

SUMMING UP: *if* $\beta = \alpha \geq 0$, *then* $h \equiv \xi$, *while if* $\beta > \alpha \geq 0$, *then* h *is convex with slope* $0 \leq h' \leq 1$, $h > (\xi - 1)^+$ *to the left of some point*

°G. Hunt [3].

$1 < c < \infty$, $h \equiv \xi - 1$ *to the right of* c, *and* $h = E_{.}[e^{-\beta t}h(x)]$ *with* $t = \min(t\colon x \geqq c)$, $x = x(t)$, *and* $e^{-\beta t}h \equiv 0$ *if* $t = \infty$.

3 Two Examples

Consider the multiplicative Brownian motion with drift $x = \exp[\sigma b + \delta t]$ with $\sigma > 0$, $b = b(t)$ a standard (additive) Brownian motion, and $-\infty < \delta < \infty$. $E_1(x) = \exp[\sigma^2/2 + \delta t]$ so $\alpha = \sigma^2/2 + \delta$. Because $h = E_{.}[e^{-\beta t}h(x)]$ with $t = \min(t : x \geqq c)$, it follows from a formula of E. B. Dynkin[3] that if G is the generator of $[x, P_{.}]$:

$$Gf(\xi) = (\sigma^2/2)\xi^2 f''(\xi) + (\sigma^2/2 + \delta)\xi f'(\xi),$$

then $Gh = \beta h$ to the left of c. Now solve for $h = (c-1)(\xi/c)^\gamma$ with an adjustable γ and find $(\sigma^2/2)\gamma^2 + \delta\gamma - \beta = 0$, or, what is the same,

$$\gamma = -\delta/\sigma^2 + \sqrt{2\beta/\sigma^2 + \delta^2/\sigma^4} > 1$$

(the negative radical is excluded). γ is actually a function of α/σ^2 and β/σ^2 alone. Besides the above formula for h, the solution requires us to locate the corner c. Consider for this purpose G expressed in terms of the new scale $d\overset{\circ}{\xi} = \xi^{1+2\delta/\sigma^2}d\xi$ and the so-called speed measure $e(d\xi) = 2\sigma^{-2}\xi^{-1+2\delta/\sigma^2}d\xi$:

$$Gf(\xi) = \frac{f^{\oplus}(d\xi)}{e(d\xi)} = \lim_{\substack{a \uparrow \xi \\ b \downarrow \xi}} \frac{f^{\oplus}(b) - f^{\oplus}(a)}{e(a, b)}$$

$$\text{with } f^{\oplus}(a) \equiv \lim_{b \downarrow a} \frac{f(b) - f(a)}{\overset{\circ}{\xi}(b) - \overset{\circ}{\xi}(a)};$$

in this language, the fact that h is excessive is expressed as $h^{\oplus}(d\xi) - \beta h e(d\xi) \leqq 0$, and computing the mass that this distribution attributes to a small neighborhood of $\xi = c$, one finds $h^{\oplus}(c) - h^{\ominus}(c) = c^{-1-2\delta/\sigma}[h^+(c) - h^-(c)] \leqq 0$. But $h^+(c) = 1$, while $h^-(c) \leqq 1$ since h is convex, so $1 = h^-(c) = (c-1)\gamma/c$, and solving for c gives $c = \gamma/(\gamma - 1)$.
The reader can easily compute all desired probabilities for this Brownian model with the help of the formulas:

$$P_\xi[x(t) \epsilon\, d\eta] = (2\pi\sigma^2 t)^{-1/2} e^{-(\lg \eta/\xi - \delta t)^2/2\sigma^2 t}\, d\eta/\eta,$$

$$P_\xi[t \,\epsilon\, dt] = (2\pi\sigma^6 t^3)^{-1/2}(\xi/c)^{-\delta/\sigma^2}\, e^{-\delta^2 t/2\sigma^2}$$
$$\lg(\xi/c)\, e^{-(\lg \xi/c)^2/2\sigma^2 t}\sigma^2 dt,$$

and

$$P_\xi[x(t) \epsilon\, d\eta, t < t] = P_\xi[x(t) \epsilon\, d\eta]$$
$$-(\eta/\xi)^{\delta/\sigma^2}\, e^{-\delta^2 t/2\sigma^2}\, (2\pi\sigma^2 t)^{-1/2}$$
$$e^{-(\lg \eta\xi/c^2)^2/2\sigma^2 t}\, d\eta/\eta;$$

in the first formula $t, \xi, \eta > 0$, while in the second and third, $t > 0$, $0 < \xi, \eta < c$.

Consider as a second example, the (multiplicative) Poisson process $x = \exp[p(\sigma t)]$; p is a standard (additive) Poisson process with jump size 1 and unit rate, *i.e.*, $P_{.}[p(t) - p(0) = n] = t^n e^{-t}/n!$. $E_1(x) = \exp[\sigma(e-1)t]$ so $\alpha = \sigma(e-1)$. Given $\xi < c$ with exit time $t = \inf(t : x \neq \xi)$ and exit place $x = x(t)$, $h(\xi) = E_\xi[e^{-\beta t}h(x)] =$

$$\int_0^\infty \sigma e^{-\sigma t}dt\, e^{-\beta t}h(e\xi) = \frac{h(e\xi)}{1 - \beta/\sigma},$$

esp., $h(\xi) = (e\xi - 1)(1 - \beta/\sigma)^{-1}$ for $c/e \leqq \xi < c$, and letting $\xi \uparrow c$ and solving for c, one finds $c = (1 - \alpha/\beta)^{-1}$. h itself is a broken line with corners at $2^{-n}c$ ($n \geqq 0$), *esp.*, $h^+(c) = 1 > h^-(c) = (\sigma + \alpha)(\sigma + \beta)^{-1}$.

4 General Warrants

Now the problem is to find *the smallest excessive function* $h \geqq (\xi - 1)^+$ *for the stopped space-time motion*

$$z(s) = [t - s, x(s)] \qquad (s \leqq t)$$
$$= [0, x(t)] \qquad (s > t),$$

i.e., the smallest function $h(t, \xi) \geqq (\xi - 1)^+$ such that $e^{-\beta s}E_\xi[h(t - s, x(s))] \uparrow h(t, \xi)$ as $s \downarrow 0$ for each $(t, \xi) \epsilon [0, \infty) \times [0, \infty)$.

1. *Define* $h^0 = (\xi - 1)^+$ *and* $h^n = \sup_{s \leqq t} e^{-\beta s}E_{.}[h^{n-1}(t - s, x(s))]$ *for* $n \geqq 1$; *then* $(\xi - 1)^+ \leqq h^n \uparrow h \leqq \xi$ *as* $n \uparrow \infty$.

PROOF as before.

2. h *is a convex function of* $\xi \geqq 0$ *with slope* $0 \leqq h^+ \leqq 1$.

PROOF as before.

3. h *is an increasing function of* $t \geqq 0$.
PROOF: h^0 is independent of $t \geqq 0$, and

$$h^n(t_2, \xi) = \sup_{s \leqq t_2} e^{-\beta s}E_\xi[h^{n-1}(t_2 - s, x(s))]$$
$$\geqq \sup_{s \leqq t_1} e^{-\beta s}E_\xi[h^{n-1}(t_1 - s, x(s))]$$
$$= h^n(t_1, \xi)$$

if h^{n-1} is an increasing function of $t \geqq 0$; now use induction and let $n \uparrow \infty$.

4. h *is the smallest (space-time) excessive function* $\geqq (\xi - 1)^+$; *it is continuous from below as function of* $t > 0$.

[3]See ref. [1].

PROOF: $h \geq e^{-\beta s}E_\cdot[h(t-s, x(s))]$ $(s \leq t)$ is obvious. Now

$$\lim_{s\downarrow 0} e^{-\beta s}E_\xi[h(t-s, x(s))] \geq h(t-,\xi) \geq (\xi-1)^+$$

for $t > 0$, and since

$$\begin{aligned} j &= (\xi-1)^+ && (t=0) \\ &= h(t-,\xi) && (t>0) \end{aligned}$$

is a (space-time) excessive function $\geq(\xi-1)^+$, it is enough to prove that h is the smallest solution $\geq(\xi-1)^+$ of $j \geq e^{-\beta s}E_\cdot[j(t-s, x(s))]$. But this is obvious.

5. $h(0+,\xi) \equiv \lim_{t\downarrow 0} h(t,\xi) = (\xi-1)^+$.

PROOF: $k(t,\xi) = E_\xi[(x(t)-1)^+] \geq (\xi-1)^+$, and since $k = E_\cdot[k(t-s, x(s))]$, $e^{-\beta s}E_\cdot[k(t-s, x(s))]$ increases to k as $s \downarrow 0$, proving $k \geq h$. Now as $t \downarrow 0$,

$$k = e^{\alpha t}\xi - 1 + E_\xi[1-x(t), x(t)<1]$$

tends to $\xi - 1$ if $\xi > 1$. But $0 \leq k(0+,\xi) = \lim_{t\downarrow 0} E_1[(\xi x - 1)^+]$ is increasing, so the proof is complete.

6. $h(\infty,\xi) = \lim_{t\uparrow\infty} h(t,\xi)$ *coincides with the perpetual warrant.*

PROOF: $h(\infty,\xi)$ is continuous (its slope falls between 0 and 1), so $e^{-\beta s}E_\xi[h(\infty, x(s))] \leq h(\infty,\xi)$ increases to $h(\infty,\xi)$ as $s \downarrow 0$, *i.e.*, $h(\infty,\xi)$ is excessive; that it is the smallest excessive function $\geq(\xi-1)^+$ is obvious.

7. $h \equiv \xi - 1$ *to the right of some point* $1 < c = c(t) < \infty$ *for* $0 \leq t \leq \infty$. *c is increasing,* $c(t-) = c$, *and* $c(\infty) < \infty$. $h > (\xi-1)^+$ *between c and* $d = d(t) < 1$. *d is decreasing,* $d(t-) = d$, *and* $d(\infty) = 0$. $h \equiv 0$ *to the left of* d. $d = e^{-t} > 0$ *if* $x = e^t$. $d \equiv 0$ *if x is a multiplicative Brownian motion.*

PROOF: use the information above and $c(\infty) < \infty$ (2.9).

8. $h(t,\xi) = E_\xi[e^{-\beta t}h(t - t, x)]$ *if t is the (space-time) exit time from the region*

$$R: 0 < s \leq t, 0 < \xi < c(s)$$

and $x = x(t)$ *is the exit place (see Figure 1 for* R *and t).*

PROOF: as before with some (mild) technical complications.

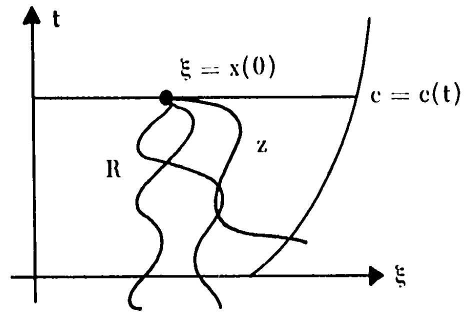

Figure 1

5 General Warrant for a Multiplicative Translation with Absorbtion

Consider the motion of translation $x = \xi \exp[(\alpha + \delta)t]$ with absorbtion at a rate $\delta \geq 0$, *i.e.*, let

$$P_1[x = \xi e^{(\alpha+\delta)t}] = e^{-\delta t} = 1 - P_1[x = 0],$$

and let us prove that

$$\begin{aligned} h(t,\xi) &= e^{-(\beta+\delta)t}[\xi e^{(\alpha+\delta)t} - 1]^+ && \xi \leq c e^{-(\alpha+\delta)t} \\ &= (\xi/c)^\gamma(c-1) && ce^{-(\alpha+\delta)t} \leq \xi \leq c \end{aligned}$$

with $\gamma = (\beta+\delta)(\alpha+\delta)^{-1}$ and $c \equiv c(\infty) = \gamma(\gamma-1)^{-1}$.

2.9 implies that the perpetual warrant is a solution of

$$e^{-(\beta+\delta)t}h[\xi e^{(\alpha+\delta)t}] = e^{-\beta t}E_\xi[h(x)] = h(\xi)$$

for $t \geq 0$ and $\xi \exp[(\alpha+\delta)t] \leq c$, or, and this is the same, a solution of

$$Gh(\xi) = \xi(\alpha+\delta)h'(\xi) - \delta h(\xi) = \beta h(\xi) \quad (\xi < c).$$

Now solve and find $h(\xi) = (\xi/c)^\gamma(c-1)$ with $\gamma = (\beta+\delta)(\alpha+\delta)^{-1}$ and an unknown corner $c \geq 1$. Because $h^-(c) \leq 1$, $(\gamma/c)(c-1) \leq 1$, while from the fact that h is excessive, it follows that

$$e^{-(\beta+\delta)t}[\xi e^{(\alpha+\delta)t} - 1] = e^{-\beta t}E_\xi[h(x)] \leq h(\xi) = \xi - 1 \quad (\xi \geq c),$$

and this cannot hold for $\xi = c$ and $t \downarrow 0$ unless $(\gamma/c)(c-1) \geq 1$, *i.e.*, unless $c = \gamma(\gamma-1)^{-1} = (\beta+\delta)(\beta-\alpha)^{-1}$.

As to the general warrant, if $\xi \geq c(t)$, then

$$e^{-(\beta+\delta)s}[\xi e^{(\alpha+\delta)s} - 1] = e^{-\beta s}E_\xi[h(t-s, x(s))] \leq h(t,\xi) = \xi - 1,$$

and solving for $\xi = c(t)$, one finds $c(t) \geq \gamma(\gamma-1)^{-1} = c = c(\infty) \geq c(t)$, *i.e.*, $c(t) \equiv c(\infty)$. Now if $c\exp[-(\alpha+\delta)t] \leq \xi \leq c$ and if $s \leq t$ is chosen so that $\xi\exp[(\alpha+\delta)s] = c$, then $h(\infty,\xi) \geq h(t,\xi) \geq e^{-\beta s}E_\xi[h(t - s, x(s))]$

$$= e^{-(\beta+\delta)s}h[t-s, e^{(\alpha+\delta)s}] = e^{-(\beta+\delta)s}(c-1) = (\xi/c)^\gamma(c-1),$$

so that $h(t, \xi) = (\xi/c)^{\gamma}(c - 1)$, while if $\xi \leq c\exp[-(\alpha + \delta)t]$, then in view of 4.8,

$$\begin{aligned} h(t, \xi) &= e^{-\beta s}E_{\xi}[h(t + s, x(s))] = \\ &\qquad e^{-(\beta+\delta)s}h(t - s, \xi\, e^{(\alpha+\delta)s}) \\ &= e^{-(\beta+\delta)t}h(0+, \xi\, e^{(\alpha+\delta)t}) \qquad (s = t-) \\ &= e^{-(\beta+\delta)t}[\xi\, e^{(\alpha+\delta)t} - 1]^{+}, \end{aligned}$$

as stated. Note that $h^{+}(t, \xi)$ jumps at $\xi \exp[(\alpha + \delta)t] = 1$ and c but not at $\xi = c$.

6 General Warrant for a Multiplicative Brownian Motion with Drift

Now let us compute as far as possible the general warrant for the multiplicative Brownian motion $x = \exp[\sigma b + \delta t]$ of 3, granting that c and the left slope $h^{-}(t, c)$ are continuous, that $c(0+) = 1$, and that c has a continuous slope $c^{\bullet}$ for $t > 0$. Consider

$$Gf(\xi) = (\sigma^2/2)\xi^2 f''(\xi) + (\sigma^2/2 + \delta)\xi f'(\xi) = f^{\oplus}(d\xi)/e(d\xi)$$

as in 3 and let us prove that h is a solution of the *free boundary problem*:

$$\begin{aligned} &(G - \partial/\partial t)h = \beta h \ \textit{on the region} \\ &\qquad R: t > 0,\ 0 < \xi < c(t) \\ &h(t, 0+) = 0 \qquad (t > 0) \\ &h(0+, \xi) = 0 \qquad (\xi \leq 1) \\ &h(t, c-) = c - 1 \qquad (t > 0) \\ &h^{-}(t, c) = 1 \qquad (t > 0). \end{aligned}$$

R (or, what is the same, the free boundary c) is unknown, and it is the extra (flux) condition $h^{-} = 1$ that makes it possible to solve for *both* R and h. 4.8 implies the partial differential equation, and the evaluations of h on the 3 sides of ∂R follow from 4.1, 4.5, and 4.7. As to the flux condition $h^{-}(t, c) = 1$,

$$\int_{[t_1, t_2)\times[a, b)} dt[h^{\oplus}(t, d\xi) - \beta h e(d\xi)] \leq \int_a^b [h(t_2, \xi) - h(t_1, \xi)]\, e(d\xi)$$

is an expression of the fact that h is (space-time) excessive, and adding up the masses that these distributions attribute to the non-overlapping boxes:

$$\frac{k-1}{n} \leq t \leq \frac{k}{n},\ c\left(\frac{k-1}{n}\right) \leq \xi \leq c\left(\frac{k}{n}\right)$$

for $k \leq m$, one finds

$$\sum_{k \leq m} \int_{\frac{k-1}{n}}^{\frac{k}{n}} dt\left[h^{\oplus}\left(t, c\left(\frac{k}{n}\right)\right) - h^{\oplus}\left(t, c\left(\frac{k-1}{n}\right)\right)\right]$$

$$\leq \sum_{k \leq m} \int_{c\left(\frac{k-1}{n}\right)}^{c\left(\frac{k}{n}\right)} \left[h\left(\frac{k}{n}, \xi\right) - h\left(\frac{k-1}{n}, \xi\right) + \beta \int_{\frac{k-1}{n}}^{\frac{k}{n}} h\,dt\right] e(d\xi)$$

$$\leq \sum_{k \leq m} \left[c\left(\frac{k}{n}\right) - c\left(\frac{k-1}{n}\right) + \frac{\beta}{n} c\left(\frac{m}{n}\right)\right] \int_{c\left(\frac{k-1}{n}\right)}^{c\left(\frac{k}{n}\right)} e(d\xi).$$

Because c is continuous, it follows on letting $n \uparrow \infty$ that

$$\int_0 dt\, c^{-1-2\delta/\sigma^2}[1 - h^{-}(t, c)] = 0,$$

and since $h^{-}(t, c) \leq 1$ is also continuous, the flux condition $h^{-} = 1$ is proved.

Now transform the free boundary problem by the substitution $h = e^{-\beta t}w(t, \sigma^{-1}[\lg \xi + \delta t])$:

$$\begin{aligned} &\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial \xi^2} \ \textit{on the region:} \\ &\qquad t > 0,\ -\infty < \xi < b \equiv \sigma^{-1}[\lg c + \delta t] \\ &w(t, -\infty) = 0 \qquad (t > 0) \\ &w(0+, \xi) = 0 \qquad (\xi \leq 0) \\ &w(t, b-) = e^{\beta t}(c - 1) \quad (t > 0) \\ &w^{-}(t, b) = e^{\beta t}c \qquad (t > 0). \end{aligned}$$

Because

$$w(t, \xi) \leq e^{\beta t}h(\infty, e^{\sigma\xi - \delta t}) = e^{\beta t}[c(\infty) - 1]c(\infty)^{-\gamma}\, e^{\gamma[\sigma\xi - \delta t]}$$

to the left of $\xi = b$, it is legitimate to take a Fourier transform $\hat{w}(t, \eta) = \int_{-\infty}^{b} e^{i\xi\eta}w(t, \xi)d\xi$.

$c(0+) = 1$ implies $\hat{w}(0+, .) \equiv 0$; this leads at once to

$$\hat{w}(t, \eta) = \int_0^t e^{-\eta(t-s)/2}e^{\beta s}\, e^{i\eta b}\left[\frac{c}{2} + \left(b^{\bullet} - \frac{i\eta}{2}\right)(c - 1)\right]ds,$$

or, what is the same,

$$\begin{aligned} &\int_0^t \frac{e^{-[\xi + b(t) - b(s)]^2/2(t-s)}}{\sqrt{2\pi(t - s)}}\, e^{\beta s} \left[\frac{c}{2} + \left[b^{\bullet} - \frac{\xi + b(t) - b(s)}{2(t - s)}\right](c - 1)\right]ds \\ &\quad = w(t, \xi + b) \quad (\xi < 0) \\ &\quad = 0 \qquad (\xi > 0), \end{aligned}$$

and from this it is possible to deduce an infinite series of integral equations for the free boundary c by a) evaluation at $\xi = 0+$, b) evaluation of the slope at $\xi = 0+$, *etc.*:

$$\text{a)}\ \frac{c-1}{2} = \int_0^t \frac{e^{-[b(t)-b(s)]^2/2(t-s)}}{\sqrt{2\pi(t-s)}} e^{-\beta(t-s)} \left[\frac{c}{2} + \left[b^{\bullet} - \frac{b(t)-b(s)}{2(t-s)}\right](c-1)\right] ds,$$

$$\text{b)}\ \frac{c}{2} = \int_0^t \frac{e^{-[b(t)-b(s)]^2/2(t-s)}}{\sqrt{2\pi(t-s)}} e^{-\beta(t-s)} \left[b^{\bullet} + \beta(c-1) - \frac{b(t)-b(s)}{2(t-s)} c\right] ds,$$

etc.

I. I. Kolodner[4] treated such free boundary problems and derived a) and b) by a more complicated method. Unfortunately, it is not possible to obtain explicit solutions, though machine computation should be feasible; as a matter of fact, even the existence and uniqueness of solutions is still unproved.

[4]See Ref. [4].

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